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Hashin-Shtrikman bounds for multiphase composites and their attainability

Liping Liu

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Abstract

This paper addresses the attainability of the Hashin-Shtrikman bounds for multiphase composites, including those of conductive materials and elastic materials. It presents a new derivation of these bounds that yield a necessary and sufficient condition for optimal microstructures. A key idea is a simple characterization of the gradient Young measures associated with optimal microstructures.

Contents

1. Introduction

Since the seminal works of Hashin & Shtrikman [16, 17], finding optimal bounds on the effective properties, with or without restriction on the volume fraction, has become one of the central problems in the theory of composites (MILTON [26]). The traditional approach of finding optimal bounds is first to derive a structure-independent bound, and then to study if this bound is attainable and if so, by what kind of structures/microstructures. The optimal bounds can in general be categorized into two types according to the methods of derivation: the Hashin-Shtrikman (HS) bounds and the translation bounds (TARTAR [34]; LURIE & CHERKAEV [23]). The known optimal microstructures are coated spheres and ellipsoids (HASHIN [15]; MILTON [25]), multi-coated spheres (LURIE & CHERKAEV [24]) and multi-rank laminations. By this approach, the G-closure problem (LURIE & CHERKAEV [23]; TARTAR [36]) for two-phase well-ordered conductive materials has been resolved (TARTAR [36]; MILTON & KOHN [27]; GRABOVSKY [12]). However, for multi-phase composites of general materials, little is known about the attainability of the best bounds which, in many situations, are the HS bounds. The main purpose of this paper is to address the attainability of the HS bounds for general multi-phase composites.

The approach of this paper is motivated by an observation that a given structure/microstructure is often optimal not only for the lower HS bound but also for the upper one, and is optimal for the lower or upper HS bounds of many different material systems. We therefore consider the problem of characterizing all the material systems for which a given structure/microstructure is optimal. This point of view appears new and is important for attacking the problem of optimal bounds.

To explore the consequence of this alternate approach, we begin with a derivation of the HS bounds for multiphase composites. The derivation is based on the fact that a solution of (2.12) can be given by the gradient of a scalar function under some hypothesis on the polarization and material. We therefore avoid the difficulties associated with the "concentration-factor" in WALPOLE [38]. This derivation explains why the common choice of trial polarizations in the HS variational principle happens to be the "right" one. More importantly, it provides us a simple characterization of structures/microstructures that attain the bounds. In a periodic setting and in terms of a simple potential problem, the main condition is that the second gradient of the potential is constant in all but one phases. (The constants in different phases can be different.) At the first sight, this condition (cf., equation (3.10)) on the attaining structures seems too restrictive to be ever satisfied by any structure. This is however deceitful. In fact the familiar constructions of coated spheres and ellipsoids, multicoated spheres and multirank laminations satisfy this condition in the sense specified in Section 4 if they indeed attain the HS bounds. In a separate publication (LIU, JAMES & LEO [22]) we present a method for constructing these special structures which we call *periodic E-inclusions*.

A periodic structure attains the HS bounds if and only if the structure is a corresponding periodic E-inclusion. Therefore, from the attainability of the lower HS bound we can infer the attainability of the upper one, and the attainability of the bounds for many different material systems. We then adapt the arguments to sequences of structures or microstructures. It has proven to be useful to describe microstructures by gradient Young measures (TARTAR [35]; BALL [6]). Based on the gradient field of a periodic E-inclusion, we define a particular form of gradient Young measures as *sequential E-inclusions* (cf., (4.2)). From the estimates implied by the HS variational principles, we show that any microstructures that attain the HS bounds if, and only if their gradient fields generate a corresponding sequential E-inclusion. From the basic relation between gradient Young measures and quasiconvex functions (KINDERLEHRER & PEDREGAL [19, 20]), we can prescribe sequential E-inclusions, and therefore attainable HS bounds in terms of quasiconex functions. From this viewpoint, the attainability of the HS bounds can be attacked by the standard approach in microstructure theory: we construct particular microstructures to find inner bounds of sequential E-inclusions, and use quasiconvex functions to find outer bounds of sequential E-inclusions.

Following the above lines, we find an outer and inner bound for sequential E-inclusions. The outer bound is obtained using the null Lagrangian $\mathbb{R}^{n\times n}_{sym} \ni X \mapsto \mathbf{m}\cdot(\mathrm{Tr}(X)X - X^2)\mathbf{m}$ for any $\mathbf{m} \in \mathbb{R}^n$. The inner bound is based on a convex property of gradient Young measures (KINDERLEHRER & PEDREGAL [19]) and the existence of periodic E-inclusions corresponding to a single matrix (LIU, JAMES & LEO [22]). From these bounds on sequential E-inclusions, we obtain an outer and inner bound on the attainable HS bounds for composites of any number of phases and in any dimensional space. We remark that the individual phases and the composites are not necessarily isotropic, though some symmetries on the softest (resp. stiffest) phase are required for the lower (resp. upper) HS bound. These attainability and non-attainability results appear new. When specialized to isotropic composites of isotropic phases, our attainability results recover what were first shown by MILTON [25].

We remark that our result is mainly on the attainability of the HS bounds. The bounds (2.37) and the dual bounds (3.1) are known for various special cases, see WALPOLE [38]; MILTON [25]; ALLAIRE & KOHN [4, 5]; and NESI [29]. Mentions should be made of the works of GRABOVSKY [13, 14] who, based on the translation method and for two-phase composites, derived a set of optimal conditions which are closely related to ours, see also ALBIN, CHERKAEV & NESI [1] for two dimensional three-phase composites and SILVESTRE [31] for the cross-property bounds.

The paper is organized as follows. In Section 2 we derive the HS bounds for multiphase composites. In Section 3 we show the equivalence between the attainability of the HS bounds by a periodic structure and the existence of a corresponding periodic E-inclusion. Note that it is not the periodic structure *per se* that is needed in our derivation. For sequences of structures or microstructures, this result remains valid if one formulates it in terms of gradient Young measures. This issue is addressed in Section 4. In Section 5, we find an outer and inner bound for sequential E-inclusions, and hence an outer and inner bound for the attainable HS bounds. Finally, in Section 6 we summarize our results and discuss the directions of generalization.

2. Hashin-Shtrikman bounds for multiphase composites

Let $\mathbb{L}$ (resp. $\mathbb{L}_{el}$) be the collection of all positive definite self-adjoint tensors (resp. all positive semi-definite linear elasticity tensors), and Ω_i $(i = 0, \cdots, N)$ be a measurable disjoint subdivision of the unit cell $Y = (0,1)^n$. We consider a periodic $(N+1)$-phase composite defined by

$$\mathbf{L}(\mathbf{x}, \mathcal{O}) = \mathbf{L}_i \qquad \text{on} \quad \Omega_i \qquad (i = 0, 1, \cdots, N), \tag{2.1}$$

where the self-adjoint and positive semi-definite tensors $\mathbf{L}_i : I\!R^{m\times n} \to I\!R^{m\times n}$ $(i = 0, \cdots, N)$, either all in $\mathbb{L}$ or all in $\mathbb{L}_{el}$, describe the material properties of individual phases, $\mathcal{O} = (\Omega_0, \cdots, \Omega_N)$ is referred to as the structure of the composite, and $\theta_i = |\Omega_i|/|Y|$ $(i = 0, \cdots, N)$ are the volume fractions.

Note that the tensors $\mathbf{L}_i$ $(i = 0, \cdots, N)$ include but are not limited to linear elasticity tensors. The benefit of this general setting is that many problems, including the conductivity problem and cross-property problem, can be treated simultaneously. For a given structure $\mathcal{O} = (\Omega_0, \cdots, \Omega_N)$ and applied average field $\mathbf{F} \in I\!R^{m\times n}$, the effective tensor $\mathbf{L}^e(\mathcal{O})$ of this composite is given by

$$\mathbf{F}\cdot\mathbf{L}^e(\mathcal{O})\mathbf{F} = \min_{\mathbf{u}\in W^{1,2}_{per}(Y, I\!R^m)} \fint_Y (\nabla\mathbf{u} + \mathbf{F})\cdot\mathbf{L}(\mathbf{x}, \mathcal{O})(\nabla\mathbf{u} + \mathbf{F})d\mathbf{x}, \tag{2.2}$$

where $\fint_V \cdot\, d\mathbf{x} = \frac{1}{\text{volume}(V)}\int_V \cdot\, d\mathbf{x}$ is the average value of the integrand in region V, and a minimizer of the right-hand side, which is unique within an additive constant and is denoted by $\mathbf{u}_\mathbf{F} \in W^{1,2}_{per}(Y, I\!R^m)$, solves the following equation

$$\begin{cases} \text{div}\big[\mathbf{L}(\mathbf{x}, \mathcal{O})(\nabla\mathbf{u}_\mathbf{F} + \mathbf{F})\big] = 0 & \text{on } Y \\ \text{periodic boundary conditions} & \text{on } \partial Y \end{cases}. \tag{2.3}$$

Equivalently, the effective tensor is given by (CHRISTENSEN [9])

$$\mathbf{L}^e(\mathcal{O})\mathbf{F} = \fint_Y \mathbf{L}(\mathbf{x}, \mathcal{O})(\nabla\mathbf{u}_\mathbf{F} + \mathbf{F})d\mathbf{x} \qquad \forall\, \mathbf{F} \in I\!R^{m\times n}. \tag{2.4}$$

In general the effective tensor $\mathbf{L}^e(\mathcal{O})$ defined by equation (2.2) or (2.4) depends on many factors, such as the shape, topology and volume fractions (to mention a few), of the structure $\mathcal{O}$. Therefore, it is often more useful to find the bounds on the effective tensors that depend only on some simple features of the structure, say, volume fractions, than to calculate the exact effective tensor.

To proceed, we introduce a few notations. Denote by $\mathcal{N}(\,\cdot\,)$ and $\mathcal{R}(\,\cdot\,)$ the null space and the range space of a self-adjoint linear mapping $(\,\cdot\,)$, respectively. For two self-adjoint linear mappings $\mathbf{T}_1$, $\mathbf{T}_2 : I\!R^{m\times n} \to I\!R^{m\times n}$, we write $\mathbf{T}_1 \geq$ (resp . $\leq$)$\mathbf{T}_2$ if $\mathbf{T}_1 - \mathbf{T}_2$ is positive (resp. negative) semi-definite. We follow the conventions $1/\infty = 0$, $1/0 = \infty$, and interpret the self-adjoint inverse $\mathbf{T}^{-1}$ of a self-adjoint positive semi-definite linear mapping $\mathbf{T} : I\!R^{m\times n} \to I\!R^{m\times n}$ as

$$Z \cdot \mathbf{T}^{-1} Z = \sup_{X \in I\!R^{m\times n}} \{2Z\cdot X - X \cdot \mathbf{T}X\} \qquad \forall\, Z \in I\!R^{m\times n}. \tag{2.5}$$

Clearly, $(\mathbf{T}^{-1})^{-1} = \mathbf{T}$ and

$$Z \cdot \mathbf{T}^{-1} Z \geq c|Z|^2 \qquad \forall\, Z \in I\!R^{m\times n}, \tag{2.6}$$

where $c > 0$ is independent of Z. Clearly, $Z \notin \mathcal{R}(\mathbf{T})$ if and only if inequality (2.6) holds for arbitrary c. In this case we write $Z \cdot \mathbf{T}^{-1} Z = \infty$.

It is useful to notice

Lemma 2.1. *Let $\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N$ be all in $\mathbb{L}$ or all in $\mathbb{L}_{el}$. Consider $(N+1)$-phase composites* (2.1) *with the effective tensor $\mathbf{L}^e(\mathcal{O})$ given by* (2.2)*. If $0 \leq \mathbf{L}_c \leq \mathbf{L}(\mathbf{x}, \mathcal{O})$ (resp. $\mathbf{L}_c \geq \mathbf{L}(\mathbf{x}, \mathcal{O}) \geq 0$), and $\theta_i = |\Omega_i|/|Y| \neq 0$ for all $i = 0, \cdots, N$, then*

$$\mathcal{N}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c) = \cap_{i=0}^N \mathcal{N}(\mathbf{L}_i - \mathbf{L}_c), \tag{2.7}$$

which is equivalent to

$$\mathcal{R}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c) = \oplus_{i=0}^N \mathcal{R}(\mathbf{L}_i - \mathbf{L}_c). \tag{2.8}$$

Proof. In the case $0 \leq \mathbf{L}_c \leq \mathbf{L}(\mathbf{x}, \mathcal{O})$, from equations (2.2) and (2.3) we have

$$\begin{aligned}\mathbf{F} \cdot (\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)\mathbf{F} = &⨍_Y (\nabla \mathbf{u}_\mathbf{F} + \mathbf{F}) \cdot (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(\nabla \mathbf{u}_\mathbf{F} + \mathbf{F}) d\mathbf{x} \\ &+ ⨍_Y \nabla \mathbf{u}_\mathbf{F} \cdot \mathbf{L}_c \nabla \mathbf{u}_\mathbf{F}\, d\mathbf{x} \geq 0.\end{aligned} \tag{2.9}$$

Thus, $\mathbf{L}^e(\mathcal{O}) \geq \mathbf{L}_c$. If $\mathbf{F} \in \mathcal{N}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)$, $\mathbf{u}_\mathbf{F} = const.$ is clearly a minimizer of the right-hand side of (2.2). Thus, equation (2.9) implies

$$0 = \sum_{i=1}^N \theta_i \mathbf{F} \cdot (\mathbf{L}_i - \mathbf{L}_c)\mathbf{F}.$$

Therefore, $\mathbf{F} \in \mathcal{N}(\mathbf{L}_i - \mathbf{L}_c)$ for all $i = 0, \cdots, N$. Conversely, if $\mathbf{F} \in \cap_{i=0}^N \mathcal{N}(\mathbf{L}_i - \mathbf{L}_c)$, then $\mathbf{L}_i\mathbf{F} = \mathbf{L}_c\mathbf{F}$ for all $i = 0, \cdots, N$. Thus, $\mathbf{u}_\mathbf{F} = const.$ solves (2.3). Therefore, $\mathbf{F} \cdot \mathbf{L}^e(\mathcal{O})\mathbf{F} = \mathbf{F} \cdot \mathbf{L}_c\mathbf{F}$, i.e., $\mathbf{F} \in \mathcal{N}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)$. We thus have proved (2.7). For equation (2.8), we only need to show that $[\cap_{i=0}^N \mathcal{N}(\mathbf{L}_i - \mathbf{L}_c)]^\perp = \oplus_{i=0}^N \mathcal{R}(\mathbf{L}_i - \mathbf{L}_c)$. This is standard. The proof for the case $0 \le \mathbf{L}_c \le \mathbf{L}(\mathbf{x}, \mathcal{O})$ is now completed. The case $\mathbf{L}_c \ge \mathbf{L}(\mathbf{x}, \mathcal{O}) \ge 0$ can be handled similarly and will not be repeated here.

We now reformulate the effective tensor through the well-known HS variational principles. Let $\mathbf{L}_c \in \mathbb{L} \cup \mathbb{L}_{el}$ be a "comparison material". We recall the classic Wiener bounds

$$\mathbf{H}_\Theta \le \mathbf{L}^e(\mathcal{O}) \le \mathbf{L}_\Theta, \tag{2.10}$$

where $\mathbf{L}_\Theta = \sum_{i=0}^N \theta_i \mathbf{L}_i$ and $\mathbf{H}_\Theta = [\sum_{i=0}^N \theta_i \mathbf{L}_i^{-1}]^{-1}$ are the arithmetic mean and harmonic mean, respectively. From (2.10), it is clear that if $0 \le \mathbf{L}_c \le \mathbf{L}(\mathbf{x}, \mathcal{O})$ (resp. $\mathbf{L}_c \ge \mathbf{L}(\mathbf{x}, \mathcal{O}) \ge 0$), then $\mathbf{L}^e(\mathcal{O}) \ge \mathbf{L}_c$ (resp . $\mathbf{L}^e(\mathcal{O}) \le \mathbf{L}_c$). Let

$$\begin{cases} \mathcal{E}^L(\mathbf{P}) = ⨍_Y \big[\nabla \mathbf{v_P} \cdot \mathbf{L}_c \nabla \mathbf{v_P} + \mathbf{P} \cdot (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)^{-1} \mathbf{P}\big] d\mathbf{x} \\ \mathcal{E}^U(\mathbf{P}) = ⨍_Y \big[- \nabla \mathbf{v_P} \cdot \mathbf{L}_c \nabla \mathbf{v_P} + \mathbf{P} \cdot (\mathbf{L}_c - \mathbf{L}(\mathbf{x}, \mathcal{O}))^{-1} \mathbf{P}\big] d\mathbf{x} \end{cases}, \tag{2.11}$$

where $\mathbf{P} \in L^2_{per}(Y, I\!R^{m\times n})$ and $\mathbf{v_P} \in W^{1,2}_{per}(Y, I\!R^m)$ satisfies

$$\begin{cases} \operatorname{div}[\mathbf{L}_c \nabla \mathbf{v_P} + \mathbf{P}] = 0 & \text{on } Y \\ \text{periodic boundary conditions} & \text{on } \partial Y \end{cases}. \tag{2.12}$$

By the divergence theorem, we have

$$0 \le ⨍_Y \nabla \mathbf{v_P} \cdot \mathbf{L}_c \nabla \mathbf{v_P} \, d\mathbf{x} = -⨍_Y \mathbf{P} \cdot \nabla \mathbf{v_P} \, d\mathbf{x} \le ⨍_Y \mathbf{P} \cdot \mathbf{L}_c^{-1} \mathbf{P} \, d\mathbf{x}. \tag{2.13}$$

In particular, the second inequality follows from

$$⨍_Y (\mathbf{P} \cdot \mathbf{L}_c^{-1} \mathbf{P} - \nabla \mathbf{v_P} \cdot \mathbf{L}_c \nabla \mathbf{v_P}) \, d\mathbf{x} = ⨍_Y (\mathbf{P} - \mathbf{L}_c \nabla \mathbf{v_P}) \cdot \mathbf{L}_c^{-1} (\mathbf{P} - \mathbf{L}_c \nabla \mathbf{v_P}) \, d\mathbf{x} \ge 0.$$

With the inverse of a positive semi-definite linear mapping interpreted as (2.5), we have

Lemma 2.2. (HS variational principles) *Let* $\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N$ *be all in* $\mathbb{L}$ *or all in* $\mathbb{L}_{el}$, $\mathbf{L}^e(\mathcal{O})$ *be given by* (2.2), $\mathbf{P}^0 \in \mathcal{R}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)$,

$$\mathcal{P}(\mathbf{P}^0) := \{\mathbf{P} \in L^2_{per}(Y, I\!R^{m\times n}) : ⨍_Y \mathbf{P}(\mathbf{x}) d\mathbf{x} = \mathbf{P}^0\}, \tag{2.14}$$

$\mathbf{u_F}$ *be given by* (2.3) *with* $\mathbf{F} = (\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1} \mathbf{P}^0$, *and*

$$\mathbf{P_F}(\mathbf{x}) = (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(\nabla \mathbf{u_F} + \mathbf{F}) \in \mathcal{P}(\mathbf{P}^0). \tag{2.15}$$

For a structure $\mathcal{O} = (\Omega_0, \cdots, \Omega_N)$, *(i) if* $\mathbf{L}(\mathbf{x},\mathcal{O}) \geq \mathbf{L}_c$ *and* $\mathbf{P} \in \mathcal{P}(\mathbf{P}^0)$ *(cf.,* (2.11)*),*

$$\mathcal{E}^L(\mathbf{P}) - \mathbf{P}^0 \cdot (\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{P}^0 \geq c \fint_Y |\mathbf{P} - \mathbf{P}_{\mathbf{F}}|^2 \, d\mathbf{x}, \tag{2.16}$$

where $c > 0$ *is independent of* $\mathbf{P}$*, and*

(ii) if $\mathbf{L}(\mathbf{x},\mathcal{O}) \leq \mathbf{L}_c$ *and* $\mathbf{P} \in \mathcal{P}(\mathbf{P}^0)$*,*

$$\mathcal{E}^U(\mathbf{P}) - \mathbf{P}^0 \cdot (\mathbf{L}_c - \mathbf{L}^e(\mathcal{O}))^{-1}\mathbf{P}^0 \geq c \fint_Y |\mathbf{P} - \mathbf{P}_{\mathbf{F}}|^2 \, d\mathbf{x}, \tag{2.17}$$

where $c > 0$ *is independent of* $\mathbf{P}$*.*

Proof. We follow Milton & Kohn [27] for the proof. Let $\mathbf{v}_{\mathbf{P}_\mathbf{F}}$ be a solution of (2.12) for $\mathbf{P}_\mathbf{F}$. Plugging (2.15) into (2.12), we verify

$$\nabla \mathbf{v}_{\mathbf{P}_\mathbf{F}} = \nabla \mathbf{u}_\mathbf{F}. \tag{2.18}$$

From $\fint_Y \nabla \mathbf{u}_\mathbf{F} \, d\mathbf{x} = 0$ and (2.4), we verify

$$\fint_Y \mathbf{P}_\mathbf{F}(\mathbf{x}) \, d\mathbf{x} = \fint_Y \mathbf{L}(\mathbf{x},\mathcal{O})(\nabla \mathbf{u}_\mathbf{F} + \mathbf{F}) \, d\mathbf{x} - \mathbf{L}_c \mathbf{F} = \mathbf{P}^0. \tag{2.19}$$

For any $\mathbf{P} \in \mathcal{P}(\mathbf{P}^0)$, by (2.18) we have

$$-\fint_Y \nabla \mathbf{v}_\mathbf{P} \cdot \mathbf{L}_c \nabla \mathbf{u}_{\mathbf{P}_\mathbf{F}} d\mathbf{x} = \fint_Y \mathbf{P} \cdot \nabla \mathbf{u}_{\mathbf{P}_\mathbf{F}} d\mathbf{x} = \fint_Y \nabla \mathbf{v}_\mathbf{P} \cdot \mathbf{P}_\mathbf{F} d\mathbf{x}. \tag{2.20}$$

We now show equation (2.16). Since $\mathbf{L}(\mathbf{x},\mathcal{O}) - \mathbf{L}_c \geq 0$, by equations (2.5)-(2.6) we have

$$\begin{aligned} c\fint_Y |\mathbf{P} - \mathbf{P}_\mathbf{F}|^2 \, d\mathbf{x} &\leq \mathcal{E}^L(\mathbf{P} - \mathbf{P}_\mathbf{F}) \\ &= \mathcal{E}^L(\mathbf{P}) + \fint_Y (\mathbf{P}_\mathbf{F} - 2\mathbf{P}) \cdot [(\mathbf{L}(\mathbf{x},\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{P}_\mathbf{F} - \nabla \mathbf{u}_{\mathbf{P}_\mathbf{F}}] \, d\mathbf{x} \\ &= \mathcal{E}^L(\mathbf{P}) + [\fint_Y (\mathbf{P}_\mathbf{F} - 2\mathbf{P}) \, d\mathbf{x}] \cdot \mathbf{F} \\ &= \mathcal{E}^L(\mathbf{P}) - \mathbf{P}^0 \cdot (\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{P}^0, \end{aligned} \tag{2.21}$$

where $c > 0$ is independent of $\mathbf{P}$, and in the first equality we have used equations (2.18), (2.20), and the fact $\fint_Y \nabla \mathbf{u}_\mathbf{F} \cdot \mathbf{L}_c \nabla \mathbf{u}_\mathbf{F} d\mathbf{x} = -\fint_Y \mathbf{P}_\mathbf{F} \cdot \nabla \mathbf{u}_\mathbf{F} d\mathbf{x}$, see (2.13).

To show equation (2.17), we notice, from $\mathbf{L}_c \geq \mathbf{L}(\mathbf{x},\mathcal{O}) \geq 0$ and equation (2.5), that for any $X \in \mathcal{R}(\mathbf{L}_c)$,

$$\begin{aligned} X \cdot (\mathbf{L}_c - \mathbf{L}(\mathbf{x},\mathcal{O}))^{-1} X - X \cdot \mathbf{L}_c^{-1} X \\ = \sup_{Z \in I\!R^{m\times n}} \{2Z \cdot X - Z \cdot (\mathbf{L}_c - \mathbf{L}(\mathbf{x},\mathcal{O}))Z\} - X \cdot \mathbf{L}_c^{-1} X \\ \geq X \cdot \mathbf{L}_c^{-1} \mathbf{L}(\mathbf{x},\mathcal{O}) \mathbf{L}_c^{-1} X, \end{aligned} \tag{2.22}$$

where the inequality follows by choosing $Z = \mathbf{L}_c^{-1}X$. Therefore, from (2.22) and the second inequality in (2.13) we have

$$\mathcal{E}^U(\mathbf{P}-\mathbf{P_F}) \geq c \fint_Y |\mathbf{P}-\mathbf{P_F}|^2 d\mathbf{x}, \tag{2.23}$$

where $c > 0$ is some constant independent of $\mathbf{P}$. Then, we obtain (2.17) by the same calculations as in (2.21). The proof of the lemma is now completed.

The classical HS bounds are obtained by choosing a comparison material $\mathbf{L}_c$ and a trial "polarization" $\mathbf{P}$. Then equations (2.16) and (2.17) yield bounds on the effective tensor $\mathbf{L}^e(\mathcal{O})$. If the "right" trial polarization $\mathbf{P}$ are chosen, the bounds depend only on the volume fractions of the structures, and in many but not all situations, are attainable.

We now present a new derivation of the HS bounds for $(N+1)$-phase composites. This derivation will provide us a necessary and sufficient condition for the bounds to be attainable. We first consider a scalar function $u \in W^{2,2}_{per}(Y)$ defined as a solution of

$$\begin{cases} \Delta u = f(\mathbf{x}) & \text{on} \quad Y \\ \text{periodic boundary conditions} & \text{on} \quad \partial Y \end{cases}, \tag{2.24}$$

where $f \in L^2_{per}(Y)$. The solvability of (2.24) requires

$$\fint_Y f(\mathbf{x}) d\mathbf{x} = 0. \tag{2.25}$$

By Fourier analysis, we can represent the second gradient of a solution of (2.24) as

$$\nabla\nabla u(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}\setminus\{0\}} \frac{\mathbf{k}\otimes\mathbf{k}\hat{f}(\mathbf{k})}{|\mathbf{k}|^2} \exp(i\mathbf{k}\cdot\mathbf{x}), \tag{2.26}$$

where $\mathcal{K} = (2\pi)\mathbb{Z}^n$ is the reciprocal lattice of $\mathbb{Z}^n$ and

$$\hat{f}(\mathbf{k}) = \fint_Y f(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x})\, d\mathbf{x}.$$

A key observation of our derivation is that for some special $\mathbf{L}_c$ and $\mathbf{P}$, a solution of (2.12) is given by the gradient of the scalar potential (2.24). To see this, let δ_{ij} $(i, j = 1, \cdots, n)$ be the components of the identity matrix $\mathbf{I}$, $m = n \geq 2$, $0 \neq a \in \mathbb{R}$, and

$$(\mathbf{L}_c)_{piqj} = \mu_1^c \delta_{ij}\delta_{pq} + \mu_2^c \delta_{pj}\delta_{iq} + \lambda^c \delta_{ip}\delta_{jq}, \quad \mathbf{P}_*(\mathbf{x}) = k_c \mathbf{I}[a + f(\mathbf{x})], \tag{2.27}$$

where $k_c = \mu_1^c + \mu_2^c + \lambda^c > 0$. Note that $\mathbf{L}_c \in \mathbb{L}$ (resp. $\mathbf{L}_c \in \mathbb{L}_{el}$) implies the constants μ_1^c, μ_2^c, λ^c satisfy

$$\mu_1^c > \mu_2^c \ \ (\text{resp. } \mu_1^c = \mu_2^c), \ \ \mu_1^c + \mu_2^c > 0 \quad \text{and} \quad \lambda^c > -\frac{\mu_1^c + \mu_2^c}{n}. \tag{2.28}$$

Again by Fourier analysis (KHACHATURYAN [18]), we can represent the gradient of a solution of (2.12) with $\mathbf{P} = \mathbf{P}_*$ in (2.27) as

$$\nabla \mathbf{v}_{\mathbf{P}_*}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}\setminus\{0\}} \frac{-\mathbf{k}\otimes\mathbf{k}\hat{f}(\mathbf{k})}{|\mathbf{k}|^2} \exp(i\mathbf{k}\cdot\mathbf{x}). \tag{2.29}$$

By comparing (2.26) with (2.29), it is clear that

$$\nabla \mathbf{v}_{\mathbf{P}_*}(\mathbf{x}) = -\nabla\nabla u(\mathbf{x}). \tag{2.30}$$

Assuming $\mathbf{I} \in \mathcal{R}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)$ and choosing $\mathbf{P}_*(\mathbf{x})$ as a trial polarization of the right-hand sides of equations (2.16) and (2.17), we obtain

$$\begin{cases} \mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I} \le \mathcal{E}^L(\mathbf{P}_*) = I_1(f) + I_2(f) & \text{if } \mathbf{L}(\mathbf{x},\mathcal{O}) \ge \mathbf{L}_c, \\ \mathbf{I}\cdot(\mathbf{L}_c - \mathbf{L}^e(\mathcal{O}))^{-1}\mathbf{I} \le \mathcal{E}^U(\mathbf{P}_*) = -I_1(f) - I_2(f) & \text{if } \mathbf{L}(\mathbf{x},\mathcal{O}) \le \mathbf{L}_c. \end{cases} \tag{2.31}$$

Here the integrals $I_1(f)$ and $I_2(f)$ are as follows:

$$I_1(f) := \frac{1}{a^2 {k_c}^2} \fint_Y \nabla \mathbf{v}_{\mathbf{P}_*} \cdot \mathbf{L}_c \nabla \mathbf{v}_{\mathbf{P}_*} d\mathbf{x} = \frac{1}{a^2 k_c} \fint_Y f(\mathbf{x})^2 \, d\mathbf{x}, \tag{2.32}$$

where the equality follows from (2.30), (2.24), and the fact that $\fint_Y |\nabla\nabla u|^2 d\mathbf{x} = \fint_Y |\Delta u|^2 d\mathbf{x}$. Also,

$$\begin{aligned} I_2(f) &:= \frac{1}{a^2} \fint_Y (a + f(\mathbf{x}))^2 \mathbf{I}\cdot(\mathbf{L}(\mathbf{x},\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I} d\mathbf{x} \\ &= \frac{1}{a^2} \sum_{i=0}^{N} \theta_i \fint_{\Omega_i} \triangle c_i (a + f(\mathbf{x}))^2 d\mathbf{x}, \end{aligned} \tag{2.33}$$

where

$$\theta_i = \frac{|\Omega_i|}{|Y|} \quad \text{and} \quad \triangle c_i = \begin{cases} \mathbf{I}\cdot(\mathbf{L}_i - \mathbf{L}_c)^{-1}\mathbf{I} & \text{if } \mathbf{L}_i \ge \mathbf{L}_c \\ -\mathbf{I}\cdot(\mathbf{L}_c - \mathbf{L}_i)^{-1}\mathbf{I} & \text{if } \mathbf{L}_c \ge \mathbf{L}_i \end{cases}. \tag{2.34}$$

Note that only the volume fractions of the structure $\mathcal{O}$ appear in $I_1(f)$ and $I_2(f)$. To achieve the best bounds, we minimize the right-hand sides of (2.31) over all admissible $f(\mathbf{x})$. That is,

$$\begin{cases} \min\{I_1(f) + I_2(f) : \ \fint_Y f(\mathbf{x}) d\mathbf{x} = 0\} & \text{if } \ \mathbf{L}(\mathbf{x},\mathcal{O}) \ge \mathbf{L}_c \\ \min\{-I_1(f) - I_2(f) : \ \fint_Y f(\mathbf{x}) d\mathbf{x} = 0\} & \text{if } \ \mathbf{L}(\mathbf{x},\mathcal{O}) \le \mathbf{L}_c \end{cases}. \tag{2.35}$$

Using the method of Lagrangian multiplier, we obtain the minimizers, which have the same form for both minimization problems, as follows:

$$f^\star(\mathbf{x}) = \sum_{i=0}^{N} p_i \chi_{\Omega_i}, \ \ p_i = a[\frac{1}{\gamma(1/k_c + \triangle c_i)} - 1], \ \ \gamma = \sum_{i=0}^{N} \frac{\theta_i}{1/k_c + \triangle c_i}. \tag{2.36}$$

where χ_{Ω_i} is the characteristic function of region Ω_i. With $f^\star(\mathbf{x})$ chosen, equation (2.31) reads

$$\begin{cases}\mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O})-\mathbf{L}_c)^{-1}\mathbf{I}\le \triangle c_* := \frac{1}{\gamma}-\frac{1}{k_c} & \text{if } \mathbf{L}(\mathbf{x},\mathcal{O})\ge \mathbf{L}_c\\ \mathbf{I}\cdot(\mathbf{L}_c-\mathbf{L}^e(\mathcal{O}))^{-1}\mathbf{I}\le -\triangle c_* := -\frac{1}{\gamma}+\frac{1}{k_c} & \text{if } \mathbf{L}(\mathbf{x},\mathcal{O})\le \mathbf{L}_c\end{cases}. \tag{2.37}$$

Note that the two bounds in (2.37) do not contradict each other since the comparison materials $\mathbf{L}_c$ for these two bounds are different for the given $\mathbf{L}_0,\cdots,\mathbf{L}_N$, and hence that numbers $\triangle c_*$, γ, k_c are not the same in these two bounds. Moreover, the bounds are structure-independent in the sense that the numbers $\triangle c_*$ depend only on the material properties $\mathbf{L}_i$ and the volume fractions θ_i of the structures $\mathcal{O}$. In the next section we study if the equality part of " $\le$ " in (2.37) can hold, and if so, for what kind of periodic structures $\mathcal{O}$.

3. A necessary and sufficient condition for the attainment of Hashin-Shtrikman bounds

The bounds in (2.37) are inconvenient since they are restrictions on $\mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O})-\mathbf{L}_c)^{-1}\mathbf{I}$. In this section we first convert the bounds (2.37) into direct restrictions on $\mathbf{L}^e(\mathcal{O})$ using the duality of convex functions. We then show a necessary and sufficient condition for the attainment of these bounds restricted to periodic structures. As far as the bounds are concerned, the restriction to periodic structures has no loss of generality, as is well-known that any effective tensor can be approximated arbitrarily well by those of periodic composites. At the same time, the condition being necessary and sufficient implies a set of material-independent parameters that relate the attainability of these bounds for different materials. Therefore, it suffices to study the restrictions on this set of material-independent parameters to describe the attainability of these bounds for all materials of the type specified below.

We now rephrase the bounds (2.37) as direct restrictions on $\mathbf{L}^e(\mathcal{O})$.

Theorem 3.1. *Consider a periodic* $(N+1)$*-phase composite defined by* (2.1) *with volume fractions* $\theta_i=|\Omega_i|/|Y|$ $(i=0,\cdots,N)$. *If* $\mathbf{L}_c,\mathbf{L}_0,\cdots,\mathbf{L}_N$ *are all in* $\mathbb{L}$ *or all in* $\mathbb{L}_{el}$, $\mathbf{L}_c$ *satisfies* (2.27) *and* (2.28), *and* $\mathbf{I}\in\mathcal{R}(\mathbf{L}^e(\mathcal{O})-\mathbf{L}_c)$ *(see Lemma 2.1 for a sufficient condition), then the effective tensor* $\mathbf{L}^e(\mathcal{O})$ *given by* (2.2) *satisfies* (2.10) *and* (2.37). *More, the bounds* (2.37) *are equivalent to*

$$\begin{cases}\mathbf{F}\cdot(\mathbf{L}^e(\mathcal{O})-\mathbf{L}_c)\mathbf{F}\ge \mathrm{Tr}(\mathbf{F})^2/\triangle c_* & \text{if } \mathbf{L}(\mathbf{x},\mathcal{O})\ge \mathbf{L}_c\\ \mathbf{F}\cdot(\mathbf{L}^e(\mathcal{O})-\mathbf{L}_c)\mathbf{F}\le \mathrm{Tr}(\mathbf{F})^2/\triangle c_* & \text{if } \mathbf{L}(\mathbf{x},\mathcal{O})\le \mathbf{L}_c\end{cases} \tag{3.1}$$

for any $\mathbf{F}\in I\!R^{n\times n}$. *Further, one of the inequalities in* (3.1) *holds as an equality for* $\mathbf{F}\in\mathcal{R}(\mathbf{L}^e(\mathcal{O})-\mathbf{L}_c)$ *with* $\mathrm{Tr}(\mathbf{F})\neq 0$ *if, and only if the corre-*

sponding inequality in (2.37) *holds as an equality. In this case, we have*

$$\frac{\mathbf{F}}{\mathrm{Tr}(\mathbf{F})} = \frac{1}{\triangle c_*}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I}. \tag{3.2}$$

Proof. We first consider the case $\mathbf{L}(\mathbf{x}, \mathcal{O}) \geq \mathbf{L}_c$. To show (2.37) implies (3.1), for any structure $\mathcal{O}$, by the fact $\mathbf{L}^e(\mathcal{O}) \geq \mathbf{L}_c$, (2.5) and (2.37) we have

$$\sup_{\mathbf{F}\in I\!R^{n\times n}} \{2\mathbf{F}\cdot\mathbf{I} - \mathbf{F}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)\mathbf{F}\} = \mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I} \leq \triangle c_*. \tag{3.3}$$

Choosing $\mathbf{F}$ with $\mathrm{Tr}(\mathbf{F}) = \triangle c_*$, we see that $\mathbf{F}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)\mathbf{F} \geq \triangle c_* = \mathrm{Tr}(\mathbf{F})^2/\triangle c_*$, which, by multiplying $\mathbf{F}$ by a such that $a\mathrm{Tr}(\mathbf{F}) = \triangle c_*$, in fact holds for any $\mathbf{F}$ with $\mathrm{Tr}(\mathbf{F}) \neq 0$. If $\mathrm{Tr}(\mathbf{F}) = 0$, the first bound in (3.1) is obvious. We therefore conclude the first bound in (3.1) for all $\mathbf{F} \in I\!R^{n\times n}$. Further, $\mathbf{F}' = (\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I}$ is a maximizer of the left-hand side of (3.3). Therefore, if the first bound in (2.37) holds as an equality, we have $\mathrm{Tr}(\mathbf{F}') = \mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I} = \triangle c_* \neq 0$, and

$$\mathbf{F}'\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)\mathbf{F}' = \mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I} = \mathrm{Tr}(\mathbf{F}')^2/\triangle c_*.$$

Thus, the first inequality in (3.1) holds as an equality for $a\mathbf{F}'$ with any $a \neq 0$, i.e., all $\mathbf{F}$ that satisfy (3.2).

Conversely, from the first bound in (3.1), choosing $\mathbf{F} = (\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I}$ we obtain the first bound in (2.37). Further, if the first bound in (3.1) holds as an equality for $\mathbf{F} \in \mathcal{R}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)$ with $\mathrm{Tr}(\mathbf{F}) \neq 0$, we have

$$\sup_{\mathbf{P}^0\in I\!R^{n\times n}} \{2\mathbf{P}^0\cdot\mathbf{F} - \mathbf{P}^0\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{P}^0\} \tag{3.4}$$
$$= \mathbf{F}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)\mathbf{F} = \mathrm{Tr}(\mathbf{F})^2/\triangle c_*.$$

Choosing $\mathbf{P}^0 = \mathrm{Tr}(\mathbf{F})\mathbf{I}/\triangle c_*$ we have

$$\frac{2\mathrm{Tr}(\mathbf{F})^2}{\triangle c_*} - \frac{\mathrm{Tr}(\mathbf{F})^2}{\triangle c_*^2}\mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I} \leq \mathrm{Tr}(\mathbf{F})^2/\triangle c_*,$$

and hence

$$\mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I} \geq \triangle c_*,$$

which, together with the first bound in (2.37), implies that $\mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I} = \triangle c_*$, and that $\mathbf{P}^0 = \mathrm{Tr}(\mathbf{F})\mathbf{I}/\triangle c_* \in \mathcal{R}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)$ is in fact a maximizer of the left-hand side of (3.4). On the other hand, the maximization problem in (3.4) admits the unique maximizer $(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)\mathbf{F}$ in $\mathcal{R}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)$, which then implies equation (3.2). Thus, we complete the proof of Theorem 3.1 for the case $\mathbf{L}(\mathcal{O}, \mathbf{x}) \geq \mathbf{L}_c$.

The case $\mathbf{L}_c \geq \mathbf{L}(\mathcal{O}, \mathbf{x})$ can be handled similarly and will not be repeated here.

Below we refer to the first and the second inequalities in (2.37), (3.1) as the *lower* and *upper* HS bounds, respectively. To achieve the best bounds and to simplify algebraic calculations, we assume the materials $(\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N)$ satisfy

$$\begin{aligned}
&\mathcal{R}(\mathbf{L}_0 - \mathbf{L}_c) \subset \{x\mathbf{I}: \ x \in I\!R\} \quad \text{and} \\
&\mathcal{R}(\mathbf{L}_i - \mathbf{L}_c) = \mathcal{R}(\mathbf{L}_c) \supset I\!R^{n\times n}_{sym} \ \text{ for } \ i = 1, 2, \cdots, N.
\end{aligned} \tag{3.5}$$

Now we are ready to state the necessary and sufficient condition for the bounds (2.37) or (3.1) being attained by periodic structures.

Theorem 3.2. *Consider a periodic $(N+1)$-phase composite* (2.1) *of $\mathbf{L}_i$ $(i = 0, \cdots, N)$ with structure $\mathcal{O} = (\Omega_0, \cdots, \Omega_N)$ and volume fractions $(\theta_0, \cdots, \theta_N)$. Assume that $\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N$, all in $\mathbb{L}$ or all in $\mathbb{L}_{el}$, satisfy equation* (3.5)*, and that*

$$(\mathbf{L}_c)_{piqj} = \mu_1^c \delta_{ij}\delta_{pq} + \mu_2^c \delta_{pj}\delta_{iq} + \lambda^c \delta_{ip}\delta_{jq}, \qquad k_c = \mu_1^c + \mu_2^c + \lambda^c \tag{3.6}$$

satisfy (2.28)*. Let*

$$\triangle c_i = \begin{cases} \mathbf{I}\cdot(\mathbf{L}_i - \mathbf{L}_c)^{-1}\mathbf{I} & \text{if } \mathbf{L}_i \geq \mathbf{L}_c \\ -\mathbf{I}\cdot(\mathbf{L}_c - \mathbf{L}_i)^{-1}\mathbf{I} & \text{if } \mathbf{L}_c \geq \mathbf{L}_i \end{cases},$$

$$\gamma = \sum_{i=0}^{N} \frac{\theta_i}{1/k_c + \triangle c_i}, \qquad \text{and} \qquad \triangle c_* = \frac{1}{\gamma} - \frac{1}{k_c}. \tag{3.7}$$

Also, for a given $\mathbf{F} \in \mathcal{R}(\mathbf{L}_c)$ with $\mathrm{Tr}(\mathbf{F}) \neq 0$, let

$$\mathbf{Q}_i = \mathbf{F} - \mathrm{Tr}(\mathbf{F})\frac{(1 + k_c\triangle c_*)}{\triangle c_*(1 + k_c \triangle c_i)}(\mathbf{L}_i - \mathbf{L}_c)^{-1}\mathbf{I} \qquad (i = 1, \cdots, N), \tag{3.8}$$

$$p_i = \mathrm{Tr}(\mathbf{Q}_i) = \frac{\mathrm{Tr}(\mathbf{F})(\triangle c_* - \triangle c_i)}{\triangle c_*(1 + k_c\triangle c_i)} \quad \text{and} \quad \theta_0 p_0 + \sum_{i=1}^{N}\theta_i p_i = 0. \tag{3.9}$$

Then the effective tensor $\mathbf{L}^e(\mathcal{O})$, given by (2.2)*, satisfies* (2.10) *and* (2.37) *or* (3.1)*. Further, the lower or upper HS bound in* (3.1) *holds as an equality for $\mathbf{F}$ if, and only if the following overdetermined problem*

$$\begin{cases} \Delta u = \sum_{i=0}^{N} p_i \chi_{\Omega_i} & \text{on } \ Y \\ \nabla\nabla u = \mathbf{Q}_i & \text{on } \ \Omega_i, \ i = 1, \cdots, N \\ \text{periodic boundary conditions} & \text{on } \ \partial Y \end{cases} \tag{3.10}$$

admits a solution $u \in W^{2,2}_{per}(Y)$.

Proof. Note that the Theorem is trivial if $\theta_0 = 1$. Below we assume $\theta_0 < 1$. From Lemma 2.1 and (3.5), we have $\mathcal{R}(\mathbf{L}_c) = \mathcal{R}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c) \ni \mathbf{I}$.

Let us first consider the lower HS bound in (3.1). From Lemma 2.2 and Theorem 3.1, equation (3.2), for a given structure $\mathcal{O}$, we infer the following statements are equivalent:

(i) The first inequality in (3.1) holds as an equality for $\mathbf{F} \in \mathcal{R}(\mathbf{L}_c) = \mathcal{R}(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)$ with $\mathrm{Tr}(\mathbf{F}) \neq 0$.
(ii) The first inequality in (2.37) holds as an equality.
(iii) Let $a = \mathrm{Tr}(\mathbf{F})/k_c \triangle c_*$ (cf., (3.2)), $\mathbf{u_F}$ and $\mathbf{F}$ satisfy (2.3), and

$$\mathbf{P_F} = (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(\nabla \mathbf{u_F} + \mathbf{F}). \tag{3.11}$$

Then $\mathbf{F} = k_c a(\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)^{-1}\mathbf{I}$, $\fint_Y \mathbf{P_F}\, d\mathbf{x} = k_c a \mathbf{I}$ (cf., (2.4)), and on Y,

$$\mathbf{P}_* = k_c \mathbf{I}(a + f^\star(\mathbf{x})) = \mathbf{P_F} = (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(\nabla \mathbf{u_F} + \mathbf{F}), \tag{3.12}$$

where (cf., (2.36) and (3.9))

$$f^\star(\mathbf{x}) = a \sum_{i=0}^{N} [\frac{1}{\gamma(1/k_c + \triangle c_i)} - 1]\chi_{\Omega_i}(\mathbf{x}) = \sum_{i=0}^{N} p_i \chi_{\Omega_i}(\mathbf{x}). \tag{3.13}$$

In particular, (i)⇔(ii) follows from Theorem 3.1. (ii)⇔(iii) follows from (3.2) and Lemma 2.2, (2.16).

Since $\mathbf{P_F} = \mathbf{P}_*$, we have $\nabla \mathbf{v}_{\mathbf{P_F}} = \nabla \mathbf{v}_{\mathbf{P}_*}$, where $\nabla \mathbf{v}_{\mathbf{P_F}}$ and $\mathbf{P_F}$ ($\nabla \mathbf{v}_{\mathbf{P}_*}$ and $\mathbf{P}_*$) satisfy equation (2.12). From equations (3.12), (2.18) and (2.30), we obtain

$$\begin{aligned} \mathbf{P_F} &= (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(\nabla \mathbf{u_F} + \mathbf{F}) = (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(\nabla \mathbf{v}_{\mathbf{P_F}} + \mathbf{F}) \\ &= (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(-\nabla\nabla u + \mathbf{F}) = \mathbf{P}_* = k_c \mathbf{I}[a + f^\star(\mathbf{x})] \quad \text{on } Y, \end{aligned} \tag{3.14}$$

where $u \in W^{2,2}_{per}(Y)$ satisfies equation (2.24) with $f(\mathbf{x})$ replaced by the $f^\star(\mathbf{x})$ in (3.13). From equations (2.1) and (3.13), equation (3.14) is equivalent to that on each Ω_i $(i = 0, 1, \cdots, N)$,

$$(\mathbf{L}_i - \mathbf{L}_c)(-\nabla\nabla u + \mathbf{F}) = k_c \mathbf{I}[a + f^\star(\mathbf{x})] = \mathbf{I}\frac{\mathrm{Tr}(\mathbf{F})(1 + k_c \triangle c_*)}{\triangle c_*(1 + k_c \triangle c_i)}\ . \tag{3.15}$$

The proof for the case of lower HS bound is complete if we show equation (3.15) with $u \in W^{2,2}_{per}(Y)$ satisfying equation (2.24) with $f^\star(\mathbf{x})$ as in (3.13) is equivalent to (3.10) with $\mathbf{Q}_i$ given by (3.8).

From equation (3.5) and the fact $\mathbf{F} \in \mathcal{R}(\mathbf{L}_c) = \mathcal{R}(\mathbf{L}_i - \mathbf{L}_c)$ for $i = 1, \cdots, N$, it follows the equivalence between equation (3.15) and the second equation in (3.10) with $\mathbf{Q}_i$ given by (3.8) on $\Omega_1, \cdots, \Omega_N$. Since $\mathcal{R}(\mathbf{L}_0 - \mathbf{L}_c) \subset \{x\mathbf{I} :\ x \in I\!R\}$, direct calculations reveal that On Ω_0, equations (3.15) is equivalent to

$$\Delta u = \mathrm{Tr}(\mathbf{F}) - \frac{\mathrm{Tr}(\mathbf{F})\triangle c_0(1 + k_c \triangle c_*)}{\triangle c_*(1 + k_c \triangle c_0)} = p_0 \qquad \text{on } \Omega_0,$$

where $\theta_0 p_0 + \sum_{i=1}^{N} \theta_i p_i = 0$, see equations (3.8) and (3.7). We have thus completed the proof for the lower HS bound in (3.1).

The proof for the upper HS bound in (3.1) is similar and will not be repeated here.

Remark 1. For a sequence of periodic structures, Theorem 3.2 also holds in certain sense. More specifically, consider the lower HS bound in (3.1) and let $\mathcal{O}^{(k)} = (\Omega_0^{(k)}, \cdots, \Omega_N^{(k)})$ be a sequence of structures such that

$$\lim_{k\to\infty} \mathbf{F}\cdot(\mathbf{L}^e(\mathcal{O}^{(k)}) - \mathbf{L}_c)\mathbf{F} = \mathrm{Tr}(\mathbf{F})^2/\triangle c_*, \tag{3.16}$$

where $\mathbf{F} \in \mathcal{R}(\mathbf{L}_c)$ with $\mathrm{Tr}(\mathbf{F}) \neq 0$, and $\lim_{k\to\infty} \theta_i^{(k)} = \lim_{k\to\infty} |\Omega_i^{(k)}|/|Y| = \theta_i$. By a similar argument as in the proof of Theorem 3.1, we can show that

$$\lim_{k\to\infty} \mathbf{I}\cdot(\mathbf{L}^e(\mathcal{O}^{(k)}) - \mathbf{L}_c)^{-1}\mathbf{I} = \triangle c_* \tag{3.17}$$

and

$$\lim_{k\to\infty} (\mathbf{L}^e(\mathcal{O}^{(k)}) - \mathbf{L}_c)\mathbf{F} = \frac{\mathrm{Tr}(\mathbf{F})}{\triangle c_*}\mathbf{I}. \tag{3.18}$$

Let $a = \mathrm{Tr}(\mathbf{F})/k_c\triangle c_*$, $\mathbf{u}_{\mathbf{F}}^{(k)}$ and $\mathbf{F}$ satisfy (2.3) with $\mathcal{O}$ replaced by $\mathcal{O}^{(k)}$,

$$\mathbf{P}_{\mathbf{F}}^{(k)} = (\mathbf{L}(\mathcal{O}^{(k)}, \mathbf{x}) - \mathbf{L}_c)(\nabla \mathbf{u}_{\mathbf{F}}^{(k)} + \mathbf{F}), \tag{3.19}$$

and (cf., (3.12))

$$\mathbf{P}_*^{(k)}(\mathbf{x}) = k_c\mathbf{I}[a + f^{(k)}(\mathbf{x})], \tag{3.20}$$

where (cf., (3.13))

$$f^{(k)}(\mathbf{x}) = a\sum_{i=0}^{N}[\frac{1}{\gamma(1/k_c + \triangle c_i)} - 1]\chi_{\Omega_i^{(k)}}(\mathbf{x}) = \sum_{i=0}^{N} p_i \chi_{\Omega_i^{(k)}}(\mathbf{x}). \tag{3.21}$$

Direct calculation shows that

$$\lim_{k\to\infty} \fint_Y \mathbf{P}_*^{(k)} d\mathbf{x} = \frac{\mathrm{Tr}(\mathbf{F})}{\triangle c_*}\mathbf{I} \quad \text{and} \quad \lim_{k\to\infty} \mathcal{E}^L(\mathbf{P}_*^{(k)}) = \frac{\mathrm{Tr}(\mathbf{F})^2}{\triangle c_*}. \tag{3.22}$$

Let

$$\mathbf{P}_1^{(k)} = \fint_Y \mathbf{P}_{\mathbf{F}}^{(k)} d\mathbf{x} - \fint_Y \mathbf{P}_*^{(k)}(\mathbf{x}) d\mathbf{x} \in \mathcal{R}(\mathbf{L}_c)$$

for all $k = 1, 2, \cdots$. From (3.18), (2.4), and (3.19), we have

$$\lim_{k\to\infty} \fint_Y \mathbf{P}_{\mathbf{F}}^{(k)} d\mathbf{x} = \frac{\mathrm{Tr}(\mathbf{F})}{\triangle c_*}\mathbf{I},$$

which, by (3.22), implies $\lim_{k\to\infty}\mathbf{P}_1^{(k)} = 0$. From the first equation in (3.17) and (2.16), we have

$$\begin{aligned}\lim_{k\to\infty} ⨍_Y |\mathbf{P}_{\mathbf{F}}^{(k)} - \mathbf{P}_*^{(k)} - \mathbf{P}_1^{(k)}|^2\, d\mathbf{x} &= \lim_{k\to\infty} ⨍_Y |\mathbf{P}_{\mathbf{F}}^{(k)} - \mathbf{P}_*^{(k)}|^2\, d\mathbf{x} \\ &= \lim_{k\to\infty} ⨍_Y |(\mathbf{L}(\mathcal{O}^{(k)}, \mathbf{x}) - \mathbf{L}_c)(\nabla \mathbf{u}_{\mathbf{F}}^{(k)} + \mathbf{F}) - \mathbf{P}_*^{(k)}|^2\, d\mathbf{x} = 0,\end{aligned} \tag{3.23}$$

Also, let $u^{(k)} \in W_{per}^{2,2}(Y)$ satisfy equation (2.24) with $f(\mathbf{x})$ replaced by $f^{(k)}(\mathbf{x}) + f_1^{(k)}$ in (3.21), where $f_1^{(k)} \in I\!R$ is such that $⨍_Y (f^{(k)} + f_1^{(k)}) d\mathbf{x} = 0$. Clearly, $\lim_{k\to\infty} f_1^{(k)} = 0$. Let $\nabla \mathbf{v}_{\mathbf{P}_*^{(k)}}$ and $\mathbf{P}_*^{(k)}$ satisfy (2.12). By equation (2.30), $\nabla \mathbf{v}_{\mathbf{P}_*^{(k)}} = -\nabla\nabla u^{(k)}$. Since $\nabla \mathbf{u}_{\mathbf{F}}^{(k)}$ and $\mathbf{P}_{\mathbf{F}}^{(k)}$ satisfy (2.12) by (2.18), we have $\nabla[\mathbf{u}_{\mathbf{F}}^{(k)} - \mathbf{v}_{\mathbf{P}_*^{(k)}}]$ satisfies (2.12) with $\mathbf{P} = \mathbf{P}_{\mathbf{F}}^{(k)} - \mathbf{P}_*^{(k)}$. From equations (2.13) and (3.23) we obtain

$$\mathbf{L}_c[\nabla \mathbf{u}_{\mathbf{F}}^{(k)} - \nabla \mathbf{v}_{\mathbf{P}_*^{(k)}}] = \mathbf{L}_c[\nabla \mathbf{u}_{\mathbf{F}}^{(k)} + \nabla\nabla u^{(k)}] \to 0 \tag{3.24}$$

strongly in $L_{per}^2(Y, I\!R^{n\times n})$. By equations (3.23), (3.24), and similar calculations as in (3.14)-(3.15), we have

$$\begin{cases} \Delta u^{(k)} = \sum_{i=0}^N p_i \chi_{\Omega_i^{(k)}} + f_1^{(k)} & \text{on } Y, \\ \lim_{k\to\infty} \int_{\Omega_i^{(k)}} |\nabla\nabla u^{(k)} - \mathbf{Q}_i|^2\, d\mathbf{x} = 0 & \text{on } \Omega_i^{(k)},\ i = 1, \cdots, N, \\ \text{periodic boundary conditions} & \text{on } \partial Y. \end{cases} \tag{3.25}$$

Conversely, if a sequence of structures $\mathcal{O}^{(k)}$ is such that equation (3.25) holds, then it follows equation (3.16). The proof of this statement is presented in the proof of Theorem 4.1.

The overdetermined problem (3.10) does not have a solution unless the structure $\mathcal{O} = (\Omega_0, \cdots, \Omega_N)$ is very special. For their analogy with ellipsoids and their extremal properties as shown in the above theorem, we call $(\Omega_1, \cdots, \Omega_N)$ a *periodic E-inclusions* corresponding to symmetric matrices $\mathbb{K} = (\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ and volume fractions $\Theta = (\theta_1, \cdots, \theta_N)$ if the overdetermined problem (3.10) admits a solution $u \in W_{per}^{2,2}(Y)$, see Liu, James & Leo [22].

It is well-known that some special microstructures, say, confocal ellipsoids and multi-rank laminations, attain the optimal bounds for many different material systems. From the viewpoint of equations (3.10) and (3.8), this corresponds to the fact that equation (3.8) has many different solutions of $(\mathbf{L}_c, \mathbf{L}_0 \cdots, \mathbf{L}_N)$ and $\mathbf{F}$ for a given periodic E-inclusion corresponding to $\mathbb{K}$ and Θ. Therefore, it is useful to know all the material systems $(\mathbf{L}_c, \mathbf{L}_0 \cdots, \mathbf{L}_N)$ and $\mathbf{F}$ for which the composites of this periodic E-inclusion attains the lower or upper HS bound in (3.1).

Corollary 3.1. *Let tensors* $\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N$*, constants* k_c, $\triangle c_*, \triangle c_0, \cdots, \triangle c_N$*, be as in Theorem 3.2, and* $(\Omega_1, \cdots, \Omega_N)$ *be a periodic E-inclusion corre-*

sponding to symmetric matrices $\mathbb{K} = (\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ *and volume fractions* $\Theta = (\theta_1, \cdots, \theta_N)$*. If* $\mathbf{L}(\mathbf{x}, \mathcal{O}) \geq \mathbf{L}_c$ (resp. $\mathbf{L}(\mathbf{x}, \mathcal{O}) \leq \mathbf{L}_c$)*, then the* $(N+1)$*-phase periodic composite* (2.1) *of this periodic E-inclusion attains the lower (*resp *. upper) HS bounds* (3.1) *for* $\mathbf{F} \in \mathcal{R}(\mathbf{L}_c)$ *with* $\mathrm{Tr}(\mathbf{F}) \neq 0$ *if, and only if*

$$\mathbf{F} = \mathbf{Q}_i + \mathrm{Tr}(\mathbf{Q}_i)\frac{1 + k_c \triangle c_*}{\triangle c_* - \triangle c_i}(\mathbf{L}_i - \mathbf{L}_c)^{-1}\mathbf{I} \qquad \forall\, i = 1, \cdots, N. \quad (3.26)$$

Proof. This is a restatement of Theorem 3.2. We only need show that (3.26) is equivalent to (3.8), which is obvious by (3.5).

From Theorem 3.2, we can relate the attainability of the HS bounds of different material systems.

Corollary 3.2. *Consider a periodic* $(N+1)$*-phase composite*

$$\mathbf{L}(\mathbf{x}, \mathcal{O}) = \mathbf{L}_i \qquad \text{on } \ \Omega_i \qquad (i = 0, 1, \cdots, N). \quad (3.27)$$

Let $(\theta_0, \cdots, \theta_N)$ *be the volume fractions. If* $(\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N)$*, all in* $\mathbb{L}$ *or all in* $\mathbb{L}_{el}$*, satisfy* $\mathbf{L}(\mathbf{x}, \mathcal{O}) \geq \mathbf{L}_c$ *(resp.* $\mathbf{L}(\mathbf{x}, \mathcal{O}) \leq \mathbf{L}_c$*),* (3.5) *and* (3.6) *for some comparison material* $\mathbf{L}_c$*, and if the effective tensor* $\mathbf{L}^e(\mathcal{O})$ *attains the lower (resp. upper) HS bound in* (3.1) *for* $\mathbf{F} \in \mathcal{R}(\mathbf{L}_c)$ *with* $\mathrm{Tr}(\mathbf{F}) \neq 0$*, then the periodic composite (cf.,* (3.27)*)*

$$\mathbf{L}'(\mathbf{x}, \mathcal{O}) = \mathbf{L}'_i \qquad \text{on } \ \Omega_i \qquad (i = 0, \cdots, N), \quad (3.28)$$

attains the corresponding lower (resp. upper) HS bounds (3.1) *for* $\mathbf{F}' \in \mathcal{R}(\mathbf{L}'_c)$ *with* $\mathrm{Tr}(\mathbf{F}') = 0$ *if* $(\mathbf{L}'_c, \mathbf{L}'_0, \cdots, \mathbf{L}'_N)$*, all in* $\mathbb{L}$ *or all in* $\mathbb{L}_{el}$*, satisfy* $\mathbf{L}'(\mathbf{x}, \mathcal{O}) \geq \mathbf{L}'_c$ *(resp.* $\mathbf{L}'(\mathbf{x}, \mathcal{O}) \leq \mathbf{L}'_c$*),* (3.5) *and* (3.6) *for some* $\mathbf{L}'_c$*, and if*

$$\begin{aligned}\mathbf{F}' = \mathbf{F} - \mathrm{Tr}(\mathbf{F})\Big\{ &\frac{(1 + k_c \triangle c_*)}{\triangle c_*(1 + k_c \triangle c_i)}(\mathbf{L}_i - \mathbf{L}_c)^{-1}\mathbf{I} \\ &+ \frac{(\triangle c_* - \triangle c_i)}{\triangle c_*(1 + k_c \triangle c_i)} \frac{1 + k'_c \triangle c'_*}{\triangle c'_* - \triangle c'_i}(\mathbf{L}'_i - \mathbf{L}'_c)^{-1}\mathbf{I}\Big\}\end{aligned} \quad (3.29)$$

for all $i = 1, \cdots, N$*. Here* $k'_c, \triangle c'_0, \cdots, \triangle c'_N, \triangle c'_*$ *are as in* (3.6)*-*(3.7) *with* $\mathbf{L}_i$ *replaced by* $\mathbf{L}'_i$ *for all* $i = c, 0, \cdots, N$*.*

Proof. From Theorem 3.2, the attainment of the lower or upper HS bound in (3.1) is equivalent to the existence of the periodic E-inclusion corresponding to

$$\mathbf{Q}_i = \mathbf{F} - \mathrm{Tr}(\mathbf{F})\frac{(1 + k_c \triangle c_*)}{\triangle c_*(1 + k_c \triangle c_i)}(\mathbf{L}_i - \mathbf{L}_c)^{-1}\mathbf{I} \qquad (i = 1, \cdots, N). \quad (3.30)$$

From Corollary 3.1 and equation (3.26), it follows that the attainment of the lower or upper HS bound in (3.1) for materials $(\mathbf{L}'_c, \mathbf{L}'_0, \cdots, \mathbf{L}'_N)$ and $\mathbf{F}'$ if

$$\mathbf{F}' = \mathbf{Q}_i + \mathrm{Tr}(\mathbf{Q}_i)\frac{1 + k'_c \triangle c'_*}{\triangle c'_* - \triangle c'_i}(\mathbf{L}'_i - \mathbf{L}'_c)^{-1}\mathbf{I} \qquad (i = 1, \cdots, N). \quad (3.31)$$

Plugging (3.30) into (3.31) yields (3.29).

4. Optimal microstructures: sequential E-inclusions

The overdetermined problem (3.10) places non-obvious restrictions on matrices $\mathbb{K}$ and volume fractions Θ for which we can find a periodic E-inclusion. For instance, it has been shown in LIU, JAMES & LEO [22] that $\mathbb{K}$ and Θ necessarily satisfy

$$\sum_{i=1}^{N}[\theta_0 \mathrm{Tr}(\mathbf{Q}_i)+\sum_{j=1}^{N}\theta_j \mathrm{Tr}(\mathbf{Q}_j)]\theta_i\mathbf{Q}_i \geq \theta_0\sum_{i=1}^{N}\theta_i\mathbf{Q}_i^2+[\sum_{i=1}^{N}\theta_i\mathbf{Q}_i]^2. \quad (4.1)$$

A natural question arises: for what $\mathbb{K}$ and Θ, can we find a corresponding periodic E-inclusion? From Theorem 3.2, an answer to this question would gives us all the attainable HS bounds (3.1) by periodic structures. Further, it is desirable to allow sequences of structures or microstructures to attain the HS bounds. In fact, the familiar construction of coated spheres and multi-rank laminations are sequences of structures that attain the bounds in a periodic setting.

To find the restrictions on $\mathbb{K}$ and Θ and to include sequences of structures, it is useful to introduce the concept of sequential E-inclusions. Before the formal definition, let us observe the following common feature of the sequence $u^{(k)}$ satisfying (3.25) and a periodic E-inclusion specified by (3.10). In the case of a periodic E-inclusion, we let $u^{(k)}=u$ with u satisfying (3.10) for all k. Using L^p estimates for the Laplace operator we see that the sequence $u^{(k)}$ is in fact bounded in $W^{2,p}_{per}(Y)$ for any $1\leq p<\infty$ since $\Delta u^{(k)}$ is bounded in $L^\infty_{per}(Y)$ (GILBARG & TRUDINGER [11], page 235). Then for an open bounded domain D, the gradient sequence $\nabla\mathbf{v}^{(k)}$ of $\mathbf{v}^{(k)}(\mathbf{x})=\nabla u^{(k)}(k\mathbf{x})/k|_D$ generates a homogeneous Young measure ν that satisfies

$$\nu=\sum_{i=1}^{N}\theta_i\delta_{\mathbf{Q}_i}+\theta_0\mu \quad \text{and} \quad \mathrm{supp}\,\mu\subset\{X\in \mathbb{R}^{n\times n}_{sym}:\mathrm{Tr}(X)=p_0\}, \quad (4.2)$$

where $(\mathbf{Q}_1,\cdots,\mathbf{Q}_N)\in[\mathbb{R}^{n\times n}_{sym}]^N$, $(\theta_0,\theta_1,\cdots,\theta_N)\in[0,1]^{N+1}$, $\sum_{i=0}^N\theta_i=1$, $\delta_{\mathbf{Q}_i}$ are the Dirac masses at $\mathbf{Q}_i$, μ is a probability measure, and $p_0\in\mathbb{R}$ is such that $\theta_0p_0+\sum_{i=1}^N\theta_i\mathrm{Tr}(\mathbf{Q}_i)=0$. Note that the Dirac masses at $\mathbf{Q}_i$ $(i=1,\cdots,N)$ arise from region $\Omega_i^{(k)}$ in (3.25) or Ω_i in (3.10), and the condition $\mathrm{Tr}(X)=p_0$ arises from $\Delta u^{(k)}=p_0-f_1^{(k)}$ on $\Omega_0^{(k)}$ in (3.25) or $\Delta u=p_0$ on Ω_0 in (3.10).

Motivated by (4.2), we now define sequential E-inclusions, see LIU, JAMES & LEO [22].

Definition 1. A **sequential E-inclusion** is a homogeneous gradient Young measure that satisfies (4.2), has zero center of mass, and is generated by a bounded sequence in $W^{1,p}(D)$ for any $1 \le p < \infty$. Let $\mathbb{K} = (\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ and $\Theta = (\theta_1, \cdots, \theta_N)$ be as in (4.2). We say such a sequential E-inclusion corresponds to matrices $\mathbb{K}$ and volume fractions Θ.

In connection with the familiar microstructures such as coated spheres and multi-rank laminations, a sequential E-inclusion singles out the gradient fields associated with optimal microstructures, which geometrically not necessarily resemble coated spheres or multi-rank laminations. This is convenient since it is the special gradient field, not the geometric or topological properties that makes a microstructure optimal. Algebraically, the matrices $(\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ associated with a sequential E-inclusion are a better set of parameters to describe the optimal microstructures since they directly relate the optimal bounds and material properties, see (3.30). Further, direct connections between optimal microstructures with gradient Young measures and quasiconvex functions make possible the use of many tools for constructing and restricting new microstructures, see Theorem 5.2 and Theorem 5.5 for such examples.

In terms of sequential E-inclusions, from Theorem 3.2 and Remark 1 we have

Theorem 4.1. *Let tensors $\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N$, volume fractions $\Theta = (\theta_1, \cdots, \theta_N)$, average field $\mathbf{F} \in \mathcal{R}(\mathbf{L}_c)$ with $\mathrm{Tr}(\mathbf{F}) \neq 0$, matrices $\mathbb{K} = (\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$, and constant $\triangle c_*$ be as in Theorem 3.2. Consider $(N+1)$-phase periodic composites* (2.1) *of $\mathbf{L}_0, \cdots, \mathbf{L}_N$ with the effective tensors $\mathbf{L}^e(\mathcal{O})$ given by* (2.2). *Then, for any structure $\mathcal{O}$ with the prescribed volume fractions, the effective tensor $\mathbf{L}^e(\mathcal{O})$ satisfies* (2.10) *and* (2.37) *or* (3.1). *Further,*

$$\inf\{\mathbf{F} \cdot (\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)\mathbf{F} : \ |\Omega_i| = \theta_i\} = \mathrm{Tr}(\mathbf{F})^2 / \triangle c_* \quad \text{if } \mathbf{L}(\mathbf{x}, \mathcal{O}) \ge \mathbf{L}_c \tag{4.3}$$

or

$$\sup\{\mathbf{F} \cdot (\mathbf{L}^e(\mathcal{O}) - \mathbf{L}_c)\mathbf{F} : \ |\Omega_i| = \theta_i\} = \mathrm{Tr}(\mathbf{F})^2 / \triangle c_* \quad \text{if } \mathbf{L}(\mathbf{x}, \mathcal{O}) \le \mathbf{L}_c \tag{4.4}$$

if, and only if there exists a sequential E-inclusion corresponding to $\mathbb{K}$ and Θ.

Proof. Let $\mathcal{O}^{(k)}$ be a minimizing (resp. maximizing) sequence of (4.3) (resp. (4.4)). From Remark 1, (3.25) and the above calculations on the gradient Young measure generated by $\mathbf{v}^{(k)}(\mathbf{x}) = \nabla u^{(k)}(k\mathbf{x})/k|_D$, it is clearly that equation (4.3) (resp . (4.4)) implies the existence of the corresponding sequential E-inclusion.

Conversely, let ν be a sequential E-inclusion corresponding to $\mathbb{K}$ and Θ. Since ν has zero center of mass and is supported on $\mathbb{R}^{n\times n}_{sym}$, by Lemma 1 in ŠVERÁK [32], we can assume that ν is generated by $\nabla\nabla\xi^{(k)}$ with $\xi^{(k)}$ being a bounded sequence in $W^{2,p}_{per}(Y)$ for any $1 \le p < \infty$. Let

$$\Omega_i^{(k)} \subset \{\mathbf{x} \in Y : \ |\nabla\nabla\xi^{(k)}(\mathbf{x}) - \mathbf{Q}_i| < 1/k\} \ \text{ for } \ i = 1, \cdots, N \tag{4.5}$$

and $\Omega_0^{(k)} = Y \setminus (\cup_{i=1}^N \Omega_i^{(k)})$. By (4.2) we can assume $\theta_i^{(k)} = |\Omega_i^{(k)}|/|Y| \to \theta_i$ as $k \to \infty$. Let (cf., (3.21))

$$f^{(k)} = \sum_{i=0}^{N} p_i \chi_{\Omega_i^{(k)}} + f_1^{(k)} \qquad \text{on } Y, \tag{4.6}$$

where $p_0, \cdots p_N$ are as in (3.9) and $f_1^{(k)} \in I\!R$ are such that $\sum_{i=0}^N p_i \theta_i^{(k)} + f_1^{(k)} = 0$. Again, we have $\lim_{k\to\infty} f_1^{(k)} = 0$. From (4.2), (4.6) and (4.5), the Young measure associated with $\Delta\xi^{(k)} - f^{(k)}$ is the Dirac mass supported at $0 \in I\!R$ a.e. on Y, and hence up to a subsequence and without relabelling, we have for any $1 \le p < \infty$ (see TARTAR [35], Proposition 2),

$$\Delta\xi^{(k)} - f^{(k)} \to 0 \text{ strongly in } L^p_{per}(Y). \tag{4.7}$$

Now consider the composites (2.1) with structures given by $\mathcal{O}^{(k)} = (\Omega_0^{(k)}, \cdots, \Omega_N^{(k)})$. Direct calculations reveal that for any comparison material $\mathbf{L}_c : I\!R^{n\times n} \to I\!R^{n\times n}$,

$$\begin{aligned} &\fint_Y (-\nabla\nabla\xi^{(k)} + \mathbf{F}) \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(-\nabla\nabla\xi^{(k)} + \mathbf{F})\, d\mathbf{x} \\ &\quad = \fint_Y (-\nabla\nabla\xi^{(k)} + \mathbf{F}) \cdot (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(-\nabla\nabla\xi^{(k)} + \mathbf{F}) d\mathbf{x} \\ &\qquad + \fint_Y \nabla\nabla\xi^{(k)} \cdot \mathbf{L}_c \nabla\nabla\xi^{(k)} d\mathbf{x} + \mathbf{F} \cdot \mathbf{L}_c \mathbf{F} \end{aligned} \tag{4.8}$$

Let $\mathbf{L}_c$ and $\triangle c_*$ be as in (4.3) or (4.4). Sending $k \to \infty$, from equations (4.5), (3.8) or (3.15), we have

$$\begin{aligned} \lim_{k\to\infty} \fint_Y (-\nabla\nabla\xi^{(k)} + \mathbf{F}) \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(-\nabla\nabla\xi^{(k)} + \mathbf{F})\, d\mathbf{x} \\ = \mathrm{Tr}(\mathbf{F})^2/\triangle c_* + \mathbf{F} \cdot \mathbf{L}_c \mathbf{F}. \end{aligned} \tag{4.9}$$

In particular, equation (4.5) and (3.15) are used to calculate the first term on the right-hand side of (4.8), and the divergence theorem is used to calculate the second term on the right-hand side of (4.8). The rest of the proof has two steps.

Step 1. We first claim

$$\lim_{k\to\infty} \fint_Y \nabla\mathbf{w} \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(-\nabla\nabla\xi^{(k)} + \mathbf{F})\, d\mathbf{x} = 0 \ \ \forall\, \mathbf{w} \in W^{1,2}_{per}(Y). \tag{4.10}$$

To show this, we let $u^{(k)} \in W^{2,2}_{per}(Y)$ satisfy $\Delta u^{(k)} = f^{(k)}$ on Y. From equation (4.7) and the L^2 estimate of the Laplace operator, we have

$$\nabla\nabla\xi^{(k)} - \nabla\nabla u^{(k)} \to 0 \quad \text{strongly in} \quad L^2_{per}(Y). \tag{4.11}$$

Further, for any $\mathbf{w} \in W^{1,2}_{per}(Y)$,

$$\begin{aligned}
&\fint_Y \nabla \mathbf{w} \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(-\nabla\nabla u^{(k)} + \mathbf{F})\, d\mathbf{x} \\
&= \fint_Y \nabla \mathbf{w} \cdot [-\mathbf{L}_c \nabla\nabla u^{(k)} + k_c \mathbf{I}(a + f^{(k)})]\, d\mathbf{x} \qquad (4.12)\\
&+ \fint_Y \nabla \mathbf{w} \cdot \{[(\mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)}) - \mathbf{L}_c)(-\nabla\nabla u^{(k)} + \mathbf{F})] - k_c \mathbf{I}(a + f^{(k)})\}\, d\mathbf{x},
\end{aligned}$$

where $a = \mathrm{Tr}(\mathbf{F})/(k_c \triangle c_*)$. The first term on the right-hand side vanishes by equations (2.12), (2.24), (2.27) and (2.30). For the second term on the right-hand side, we notice from the proof of Theorem 3.2, (3.15) that

$$(\mathbf{L}_i - \mathbf{L}_c)(-\mathbf{Q}_i + \mathbf{F}) = k_c \mathbf{I}(a + \mathrm{Tr}(\mathbf{Q}_i)) \qquad \forall\, i = 1, \cdots, N.$$

By equations (4.5) and (4.11), we have

$$(\mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)}) - \mathbf{L}_c)(-\nabla\nabla u^{(k)} + \mathbf{F}) - k_c \mathbf{I}(a + f^{(k)}) \to 0$$

strongly in $L^2_{per}(Y, I\!R^{n\times n})$, which, together with (4.11) and (4.12), completes the proof of equation (4.10).

Step 2. Let $\mathbf{u}^{(k)}_{\mathbf{F}} \in W^{1,2}_{per}(Y)$ be a solution of (2.3) for $\mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})$ and $\mathbf{F}$. The weak form of (2.3) reads

$$\fint_Y \nabla \mathbf{w} \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(\nabla \mathbf{u}^{(k)}_{\mathbf{F}} + \mathbf{F})\, d\mathbf{x} = 0 \;\; \forall\, \mathbf{w} \in W^{1,2}_{per}(Y). \qquad (4.13)$$

Subtracting (4.13) from (4.10), we obtain

$$\lim_{k\to\infty} \fint_Y \nabla \mathbf{w} \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(\nabla\nabla \xi^{(k)} + \nabla \mathbf{u}^{(k)}_{\mathbf{F}})\, d\mathbf{x} = 0 \;\; \forall\, \mathbf{w} \in W^{1,2}_{per}(Y). \quad (4.14)$$

Since $\nabla\nabla\xi^{(k)} + \nabla \mathbf{u}^{(k)}_{\mathbf{F}}$ and $\mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(\nabla\nabla\xi^{(k)} + \nabla \mathbf{u}^{(k)}_{\mathbf{F}})$ are bounded in $L^2_{per}(Y, I\!R^{n\times n})$, up to a subsequence and without relabelling, we have

$$\begin{cases} \nabla\nabla\xi^{(k)} + \nabla \mathbf{u}^{(k)}_{\mathbf{F}} \rightharpoonup \nabla \mathbf{u}_\infty \\ \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(\nabla\nabla\xi^{(k)} + \nabla \mathbf{u}^{(k)}_{\mathbf{F}}) \rightharpoonup \mathbf{S} \end{cases} \text{weakly in } L^2_{per}(Y, I\!R^{n\times n}), \quad (4.15)$$

where $\mathbf{u}_\infty \in W^{1,2}_{per}(Y; I\!R^n)$, and $\mathbf{S} \in L^2_{per}(Y, I\!R^{n\times n})$ satisfies

$$\fint_Y \nabla \mathbf{w} \cdot \mathbf{S}\, d\mathbf{x} = 0 \;\; \forall\, \mathbf{w} \in W^{1,2}_{per}(Y).$$

From (4.14), (4.15), and the Div-Curl Lemma (MURAT [28]; TARTAR [34]), we have

$$\lim_{k\to\infty} \fint_Y (\nabla\nabla\xi^{(k)} + \nabla \mathbf{u}_{\mathbf{F}}^{(k)}) \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(\nabla\nabla\xi^{(k)} + \nabla \mathbf{u}_{\mathbf{F}}^{(k)})\, d\mathbf{x} = \fint_Y \nabla \mathbf{u}_\infty \cdot \mathbf{S} = 0.$$

Since $(\mathbf{L}_0, \cdots, \mathbf{L}_N)$ are either all in $\mathbb{L}$ or all in $\mathbb{L}_{el}$, we have

$$\mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})[\nabla\nabla\xi^{(k)} + \nabla \mathbf{u}_{\mathbf{F}}^{(k)}] \to 0 \text{ strongly in } L^2_{per}(Y, I\!R^{n\times n}).$$

Therefore, the effective tensors of $\mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})$ satisfy

$$\begin{aligned}\lim_{k\to\infty} \mathbf{F}\cdot \mathbf{L}^e(\mathcal{O}^{(k)})\mathbf{F} &= \lim_{k\to\infty} \fint_Y (\nabla \mathbf{u}_{\mathbf{F}}^{(k)} + \mathbf{F}) \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(\nabla \mathbf{u}_{\mathbf{F}}^{(k)} + \mathbf{F})\, d\mathbf{x} \\ &= \lim_{k\to\infty} \fint_Y (\nabla\nabla\xi^{(k)} - \mathbf{F}) \cdot \mathbf{L}(\mathbf{x}, \mathcal{O}^{(k)})(\nabla\nabla\xi^{(k)} - \mathbf{F})\, d\mathbf{x} \\ &= \mathbf{F}\cdot \mathbf{L}_c \mathbf{F} + \mathrm{Tr}(\mathbf{F})^2/\triangle c^*,\end{aligned}$$

where equation (4.9) has been used in the last equality. Thus, the bounds (4.3) or (4.4) are attained by the sequence of structures $\mathcal{O}^{(k)}$. The proof of the theorem is now completed.

5. Applications

5.1. An outer bound for sequential E-inclusions

From the basic relation between gradient Young measures and quasiconvex functions (KINDERLEHRER & PEDREGAL [19, 20]), we have

Theorem 5.1. *Let* $(\mathbf{Q}_1, \cdots, \mathbf{Q}_N) \in [I\!R^{n\times n}_{sym}]^N$ *and* $(\theta_0, \cdots, \theta_N) \in [0,1]^{N+1}$. *Let* ν *be a probability measure with zero center of mass satisfying*

$$\nu = \sum_{i=1}^N \theta_i \delta_{\mathbf{Q}_i} + \theta_0 \mu,$$

where μ *is a probability measure with* $\mathrm{supp}\,\mu \subset \{X \in I\!R^{n\times n}_{sym} : \mathrm{Tr}(X) = p_0\}$. *Then* ν *is a sequential E-inclusion if, and only if*

$$\int_{I\!R^{n\times n}} \psi(X)\, d\nu(X) = \sum_{i=1}^N \theta_i \psi(\mathbf{Q}_i) + \theta_0 \int_{I\!R^{n\times n}} \psi(X)\, d\mu(X) \ge \psi(0) \quad (5.1)$$

for all quasiconvex functions $\psi : I\!R^{n\times n} \to I\!R$ *satisfying* $|\psi(X)| \le C(|X|^p + 1)$ *for some* $C > 0$ *and* $1 \le p < \infty$.

As a first application of Theorem 5.1, we show equation (4.1) is also a restriction on sequential E-inclusions. Since a sequential E-inclusion ν is supported on symmetric matrices, inequality (5.1) is valid for quasiconvexity functions restricted to $I\!R^{n\times n}_{sym}$, see Corollary 1 in ŠVERÁK [32]. It is easy to show that the quadratic function $\psi(X) = \mathbf{m}\cdot(\mathrm{Tr}(X)X - X^2)\mathbf{m}$ is convex on symmetric rank-one matrices and therefore quasiconvex restricted to $I\!R^{n\times n}_{sym}$ for any $\mathbf{m} \in I\!R^n$. (In fact, ψ is a null Lagrangian in this second gradient context.) An application of (5.1) to $\pm\psi$ yields equation (4.1), see details in LIU, JAMES & LEO [22].

To see the implication of equation (4.1) on the attainable HS bounds, we define the following hyper-surfaces

$$\begin{cases} \mathcal{S}^L = \{\,\hat{\mathbf{L}} \in \mathbb{L} \,:\, \mathbf{I}\cdot(\,\hat{\mathbf{L}} - \mathbf{L}_c)^{-1}\mathbf{I} = \triangle c_*,\ \hat{\mathbf{L}} \ge \mathbf{L}_c\} \\ \mathcal{S}^U = \{\,\hat{\mathbf{L}} \in \mathbb{L} \,:\mathbf{I}\cdot(\mathbf{L}_c - \,\hat{\mathbf{L}})^{-1}\mathbf{I} = -\triangle c_*,\ \hat{\mathbf{L}} \le \mathbf{L}_c\} \end{cases}, \tag{5.2}$$

where constants $\triangle c_*$ are as in (2.37). Similarly, $\mathcal{S}^L_{el}$ and $\mathcal{S}^U_{el}$ can be defined by requiring $\hat{\mathbf{L}} \in \mathbb{L}_{el}$. Recall that G_Θ-closure of composites of material $(\mathbf{L}_0, \cdots, \mathbf{L}_N)$, denoted by $G_\Theta(\mathbf{L}_0, \cdots, \mathbf{L}_N)$, is the closure of all effective tensors that can be achieved by a composite of $(\mathbf{L}_0, \cdots, \mathbf{L}_N)$ with volume fractions $(\theta_0, \cdots, \theta_N)$. From (4.1) and Theorem 4.1, we have

Theorem 5.2. *Let tensors $\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N$, volume fractions $\Theta = (\theta_1, \cdots, \theta_N)$, and constant k_c, $\triangle c_*$, $\triangle c_0, \cdots, \triangle c_N$ be as in Theorem 3.2. For a given $\hat{\mathbf{L}} \in \mathbb{L} \cup \mathbb{L}_{el}$, we define*

$$\mathbf{Q}_i = [\frac{1}{\triangle c_*}(\,\hat{\mathbf{L}} - \mathbf{L}_c)^{-1} - \frac{(1 + k_c\triangle c_*)}{\triangle c_*(1 + k_c\triangle c_i)}(\mathbf{L}_i - \mathbf{L}_c)^{-1}]\mathbf{I} \tag{5.3}$$

for $i = 1, \cdots, N$. If $\mathbf{L}_i \in \mathbb{L}\,(\mathrm{resp}\,.\ \mathbb{L}_{el})$ and $\mathbf{L}_i \ge \mathbf{L}_c$ for all $i = 0, \cdots, N$, and if $\hat{\mathbf{L}} \in \mathcal{S}^L(\mathrm{resp}\,.\ \mathcal{S}^L_{el})$ is such that $\mathbb{K} = (\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ defined by (5.3) and Θ do not satisfy (4.1), then $\hat{\mathbf{L}}$ is not attainable in the sense that there is no sequence of structures $\mathcal{O}^{(k)} = (\Omega_0^{(k)}, \cdots, \Omega_N^{(k)})$ satisfying $\lim_{k\to\infty}|\Omega_i^{(k)}|/|Y| = \theta_i$ $(i = 1, \cdots, N)$ such that

$$\lim_{k\to\infty} \mathbf{L}^e(\mathcal{O}^{(k)}) = \hat{\mathbf{L}}. \tag{5.4}$$

In another word, $\hat{\mathbf{L}} \notin G_\Theta(\mathbf{L}_0, \cdots, \mathbf{L}_N)$.

It is clear that the analogous result holds for the upper bound, which will not be repeated here.

Proof. Assume there exists a sequence of structure $\mathcal{O}^{(k)}$ with the prescribed volume fractions such that (5.4) holds. Since $\hat{\mathbf{L}} \in \mathcal{S}^L(\text{or } \mathcal{S}^L_{el})$, $\mathbf{F} = (\,\hat{\mathbf{L}} - \mathbf{L}_c)^{-1}\mathbf{I}$ with $\mathrm{Tr}(\mathbf{F}) \ne 0$ is well-defined if $\triangle c_* < \infty$. Direct calculations reveal that $\mathcal{O}^{(k)}$ is a minimizing sequence such that (4.3) holds for this $\mathbf{F}$, and hence by Theorem 4.1 (see also Remark 1), we have the existence of the corresponding sequential E-inclusion with matrices $\mathbb{K} = (\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ given by (5.3) and volume fractions Θ. This however contradicts (4.1) by assumption.

We remark that Theorem 5.2 gives an outer bound on the attainable HS bounds (2.37) or (3.1). Although it is algebraically tedious to verify (4.1), we emphasize that this non-attainability result is nontrivial if $N \geq 2$, see discussions in Section 5.3. Of course, there are many other quasiconvex functions that could be used in equation (5.1) that would evidently give further restrictions on the $\mathbb{K}$ and Θ, and therefore by Theorem 4.1, give further restrictions on the attainable HS bounds.

5.2. An inner bound for sequential E-inclusions

In this section we derive an inner bound for sequential E-inclusions, which then implies an inner bound on the attainable HS bounds. For brevity, we refer to a gradient Young measure that is generated by a bounded sequence in $W^{1,p}(D)$ as a $W^{1,p}$ *gradient Young measure*, see KINDERLEHRER & PEDREGAL [20]. Similarly, a sequential E-inclusion is a $W^{1,\infty}$ sequential E-inclusion if it can be generated by a bounded sequence in $W^{1,\infty}(D)$. Recall the following two theorems:

Theorem 5.3. *(*Theorem 3.1, KINDERLEHRER & PEDREGAL [19]*) Let ν_1 and ν_2 be two homogeneous $W^{1,\infty}$ gradient Young measures with zero center of mass. Then for each $\lambda \in [0,1]$, the measure $(1-\lambda)\nu_1 + \lambda\nu_2$ is also a homogeneous $W^{1,\infty}$ gradient Young measure with zero center of mass.*

Theorem 5.4. *(*Theorem 3, LIU, JAMES & LEO [22] *) Let $\mathbf{Q} \in I\!R^{n\times n}_{sym}$ be either negative semi-definite or positive semi-definite. Then for each $\theta \in [0,1]$, corresponding to $\mathbf{Q}$ and θ, there exists a $W^{1,\infty}$ sequential E-inclusion*

$$\nu = \theta\delta_{\mathbf{Q}} + (1-\theta)\mu, \tag{5.5}$$

where μ is a probability measure and $\operatorname{supp}\mu \subset \{X \in I\!R^{n\times n}_{sym} : \theta\mathrm{Tr}(\mathbf{Q}) + (1-\theta)\mathrm{Tr}(X) = 0\}$.

From Theorem 5.4 and Theorem 5.3, we have

Theorem 5.5. *Let $\mathbb{K} = (\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ be either all negative semi-definite or all positive semi-definite, and $\Theta = (\theta_1, \cdots, \theta_N)$ be any array satisfying*

$$\theta_1, \cdots, \theta_N \in [0,1], \qquad \theta_0 = 1 - \sum_{i=1}^{N} \theta_i \in [0,1]. \tag{5.6}$$

Then corresponding to $\mathbb{K}$ and Θ, there exists a $W^{1,\infty}$ sequential E-inclusion

$$\nu = \theta_i \sum_{i=1}^{N} \delta_{\mathbf{Q}_i} + \theta_0\mu, \tag{5.7}$$

where μ is a probability measure and $\operatorname{supp}\mu \subset \{X \in I\!R^{n\times n}_{sym} : \sum_{i=1}^{N} \theta_i \mathrm{Tr}(\mathbf{Q}_i) + \theta_0\mathrm{Tr}(X) = 0\}$.

Proof. We prove the theorem by induction. If $N = 1$, the theorem holds by Theorem 5.4. Assume the theorem holds for $1 \le N \le k$, below we show the theorem holds for $N = k + 1$.

Let $\Theta = (\theta_1, \cdots, \theta_{k+1})$ satisfy (5.6) for $N = k + 1$. By multiplying the generating sequence $\mathbf{v}^{(k)}$ by any constant $a \in I\!R$, we see that there exists a $W^{1,\infty}$ sequential E-inclusions corresponding to $(a\mathbf{Q}_1, \cdots, a\mathbf{Q}_{k+1})$ and Θ if there exists a $W^{1,\infty}$ sequential E-inclusions corresponding to $(\mathbf{Q}_1, \cdots, \mathbf{Q}_{k+1})$ and Θ. Therefore, without loss of generality we may assume $\mathbb{K} = (\mathbf{Q}_1, \cdots, \mathbf{Q}_{k+1})$ are all negative semi-definite. Let $p_0 \in I\!R$ be such that $\theta_0 p_0 + \sum_{i=1}^{k+1} \theta_i \mathrm{Tr}(\mathbf{Q}_i) = 0$ and $\alpha \in [0, 1]$ be such that $\alpha \mathrm{Tr}(\mathbf{Q}_{k+1}) + (1 - \alpha) p_0 = 0$. If any $\theta_i = 0$ or $p_0 = 0$, the theorem for $N = k+1$ follows trivially from the inductive assumption. In particular, if $\theta_0 = 0$ or $p_0 = 0$, from the negative semi-definiteness of $(\mathbf{Q}_1, \cdots, \mathbf{Q}_{k+1})$, the only possible form of ν in (5.7) is the Dirac measure supported at the zero matrix. We subsequently assume $\theta_i \in (0, 1)$ for all $i = 0, \cdots, k + 1$ and $p_0 > 0$.

It is easy to verify that

$$\alpha = \frac{p_0}{p_0 - \mathrm{Tr}(\mathbf{Q}_{k+1})} > \frac{p_0 - p_0 \sum_{i=1}^{k} \theta_i + \sum_{i=1}^{k} \theta_i \mathrm{Tr}(\mathbf{Q}_i)}{p_0 - \mathrm{Tr}(\mathbf{Q}_{k+1})} = \theta_{k+1}, \quad (5.8)$$

and that

$$\begin{aligned} &\theta_{k+1} \mathrm{Tr}(\mathbf{Q}_{k+1}) + \theta_0 p_0 \ge 0 \implies \\ &\qquad \frac{\theta_{k+1}}{\alpha}(1 - \alpha) = \frac{-\mathrm{Tr}(\mathbf{Q}_{k+1})\theta_{k+1}}{p_0} \le \theta_0. \end{aligned} \quad (5.9)$$

Define λ and θ'_i $(i = 1, \cdots, k)$ by

$$\lambda = \frac{\theta_{k+1}}{\alpha}, \quad (1 - \lambda)\theta'_0 + \lambda(1 - \alpha) = \theta_0 \quad \text{and} \quad (1 - \lambda)\theta'_i = \theta_i \ . \quad (5.10)$$

From equations (5.8) and (5.9), we see that $\lambda \in (0, 1)$ and $\theta'_0, \cdots, \theta'_k \ge 0$. In particular, $\theta'_0 \ge 0$ follows from (5.9) and $(1 - \lambda)\theta'_0 = \theta_0 - \lambda(1 - \alpha) = \theta_0 - \frac{\theta_{k+1}}{\alpha}(1 - \alpha)$. Further,

$$\sum_{i=0}^{k} \theta'_i = \frac{1}{1 - \lambda} \sum_{i=0}^{k} \theta_i - \frac{\lambda(1 - \alpha)}{1 - \lambda} = \frac{-\lambda + \sum_{i=0}^{k+1} \theta_i}{1 - \lambda} = 1.$$

Thus, $(\theta'_1, \cdots, \theta'_k)$ satisfy (5.6) for $N = k$. By the inductive assumption, for $N = k$ we have the existence of a $W^{1,\infty}$ sequential E-inclusion

$$\nu_1 = \sum_{i=1}^{k} \theta'_i \delta_{\mathbf{Q}_i} + \theta'_0 \mu_1, \quad (5.11)$$

where μ_1 is a probability measure with

$$\operatorname{supp}\mu_1 \subset \{X \in I\!R^{n\times n}_{sym} :\ \sum_{i=1}^{k} \theta_i' \mathrm{Tr}(\mathbf{Q}_i) + \theta_0' \mathrm{Tr}(X) = 0\}.$$

By Theorem 5.4, we also have the existence of a $W^{1,\infty}$ sequential E-inclusion

$$\nu_2 = \alpha\delta_{\mathbf{Q}_{k+1}} + (1-\alpha)\mu_2, \tag{5.12}$$

where μ_2 is a probability measure with

$$\operatorname{supp}\mu_2 \subset \{X \in I\!R^{n\times n}_{sym} :\ \mathrm{Tr}(X) = p_0\}.$$

Let p_0' be such that $\sum_{i=1}^{k} \theta_i' \mathrm{Tr}(\mathbf{Q}_i) + \theta_0' p_0' = 0$. From equation (5.10), we have $\sum_{i=1}^{k} \frac{\theta_i \mathrm{Tr}(\mathbf{Q}_i)}{1-\lambda} + \frac{p_0'}{1-\lambda}[\theta_0 - \lambda(1-\alpha)] = 0$, which, by equation (5.9) and the definition of p_0, implies

$$\begin{aligned} 0 &= [\theta_0 + \frac{\theta_{k+1}\mathrm{Tr}(\mathbf{Q}_{k+1})}{p_0}]p_0' - \theta_{k+1}\mathrm{Tr}(\mathbf{Q}_{k+1}) - p_0\theta_0 \\ &= \frac{1}{p_0}[\theta_0 p_0 + \theta_{k+1}\mathrm{Tr}(\mathbf{Q}_{k+1})](p_0' - p_0). \end{aligned} \tag{5.13}$$

If $\theta_0 p_0 + \theta_{k+1}\mathrm{Tr}(\mathbf{Q}_{k+1}) = 0$, by equations (5.9) and (5.10) we have $\theta_0' = 0$. If $\theta_0 p_0 + \theta_{k+1}\mathrm{Tr}(\mathbf{Q}_{k+1}) \neq 0$, by equation (5.13) we have $p_0 = p_0'$. For either case, define

$$\nu := \lambda\nu_2 + (1-\lambda)\nu_1 = \sum_{i=1}^{k+1} \theta_i \delta_{\mathbf{Q}_i} + \theta_0 \mu, \tag{5.14}$$

where $\mu = \frac{\lambda(1-\alpha)}{\theta_0}\mu_2 + \frac{(1-\lambda)\theta_0'}{\theta_0}\mu_1$ is a probability measure with supp $\mu \subset \{X \in I\!R^{n\times n}_{sym} :\ \mathrm{Tr}(X) = p_0\}$. From Theorem 5.3 and Definition 1, we see that ν in (5.14) is a $W^{1,\infty}$ sequential E-inclusion corresponding to $\mathbb{K}$ and Θ. The proof of the theorem is completed.

From Theorem 4.1 and Theorem 5.5, we have the following sufficient conditions for the HS bounds to be attainable.

Theorem 5.6. *Let tensors $\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N$, volume fractions $(\theta_0, \cdots, \theta_N)$, and constants $k_c, \triangle c_1 \cdots, \triangle c_N$, $\triangle c_*$ be as in Theorem 3.2. Consider $(N+1)$-phase periodic composites* (2.1) *of $\mathbf{L}_0, \cdots, \mathbf{L}_N$ with the effective tensors $\mathbf{L}^e(\mathcal{O})$ given by* (2.2)*. Then the effective tensors satisfy the HS bounds* (2.10) *and* (2.37) *or* (3.1)*. The bounds in* (3.1) *are attainable for $\mathbf{F} \in \mathcal{R}(\mathbf{L}_c)$ with* $\mathrm{Tr}(\mathbf{F}) \neq 0$ *if the matrices*

$$\mathbf{Q}_i = \mathbf{F} - \mathrm{Tr}(\mathbf{F})\frac{(1 + k_c\triangle c_*)}{\triangle c_*(1 + k_c \triangle c_i)}(\mathbf{L}_i - \mathbf{L}_c)^{-1}\mathbf{I} \qquad (i = 1, \cdots, N) \tag{5.15}$$

are all symmetric and either all negative semi-definite or all positive semi-definite.

Below we illustrate the applications of Theorem 5.2 and Theorem 5.6 by two examples.

5.3. Composites of conductive materials

For the first example, let us consider conductive composites of $(N+1)$-phases with conductivity tensors $0 < \mathbf{A}_0, \cdots, \mathbf{A}_N \in I\!R^{n\times n}_{sym}$ and volume fractions $\theta_0 \in [0,1]$, $\Theta = (\theta_1, \cdots, \theta_N) \in [0,1]^N$. The effective conductivity tensor of a composite is denoted by $\mathbf{A}^e$. In accordance with (3.5), we assume

$$\mathbf{A}_0 = k_0\mathbf{I}, \quad \mathbf{A}_N = k_N\mathbf{I}, \tag{5.16}$$

and

$$\mathbf{A}_0 < \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_{N-1} < \mathbf{A}_N. \tag{5.17}$$

To use Theorem 3.1, we set $(\mathbf{L}_i)_{pjqk} = \delta_{pq}(\mathbf{A}_i)_{jk}$ for $i = 0, \cdots, N$ and choose $(\mathbf{L}_c)_{pjqk} = k_0\delta_{pq}\delta_{jk}$ for the lower bound and $(\mathbf{L}_c)_{pjqk} = k_N\delta_{pq}\delta_{jk}$ for the upper bound. With the effective tensors $\mathbf{L}^e$ defined as in (2.2), one can easily verify that they can be written as $(\mathbf{L}^e)_{piqj} = \delta_{pq}(\mathbf{A}^e)_{ij}$. By (2.10) and (5.17), we have

$$\mathbf{A}_0 \le \mathbf{H}_\Theta \le \mathbf{A}^e \le \mathbf{A}_\Theta \le \mathbf{A}_N, \tag{5.18}$$

where $\mathbf{A}_\Theta = \sum_{i=0}^N \theta_i\mathbf{A}_i$ and $\mathbf{H}_\Theta = [\sum_{i=0}^N \theta_i\mathbf{A}_i^{-1}]^{-1}$ are the arithmetic mean and harmonic mean, respectively. From equations (2.37) and (3.1), we have

$$\begin{cases} \mathrm{Tr}[(\mathbf{A}_0^{-1/2}\mathbf{A}^e\mathbf{A}_0^{-1/2} - \mathbf{I})^{-1}] \le \triangle c_*^L \\ \mathrm{Tr}[(\mathbf{I} - \mathbf{A}_N^{-1/2}\mathbf{A}^e\mathbf{A}_N^{-1/2})^{-1}] \le -\triangle c_*^U \end{cases}, \tag{5.19}$$

and

$$\begin{cases} \mathrm{Tr}(\mathbf{F}^T\mathbf{F}) + \mathrm{Tr}(\mathbf{F})^2/\triangle c_*^L \le \mathbf{A}_0^{-1/2}\mathbf{A}^e\mathbf{A}_0^{-1/2} \cdot \mathbf{F}^T\mathbf{F} \\ \mathrm{Tr}(\mathbf{F}^T\mathbf{F}) + \mathrm{Tr}(\mathbf{F})^2/\triangle c_*^U \ge \mathbf{A}_N^{-1/2}\mathbf{A}^e\mathbf{A}_N^{-1/2} \cdot \mathbf{F}^T\mathbf{F} \end{cases} \quad \forall\, \mathbf{F} \in I\!R^{n\times n}, \tag{5.20}$$

where, by (3.7),

$$\triangle c_*^L = \frac{\sum_{i=0}^N \theta_i \triangle c_i^L/(1+\triangle c_i^L)}{\sum_{i=0}^N \theta_i/(1+\triangle c_i^L)}, \tag{5.21}$$

$$\triangle c_*^U = \frac{\sum_{i=0}^N \theta_i \triangle c_i^U/(1+\triangle c_i^U)}{\sum_{i=0}^N \theta_i/(1+\triangle c_i^U)}, \tag{5.22}$$

and

$$\begin{cases} \triangle c_i^L = \mathrm{Tr}[(\mathbf{A}_0^{-1/2}\mathbf{A}_i\mathbf{A}_0^{-1/2} - \mathbf{I})^{-1}] > 0 \\ \triangle c_i^U = -\mathrm{Tr}[(\mathbf{I} - \mathbf{A}_N^{-1/2}\mathbf{A}_i\mathbf{A}_N^{-1/2})^{-1}] < 0 \end{cases} \quad (i = 0, \cdots, N). \tag{5.23}$$

We remark that through a linear transformation, one can show that the bounds (5.18), (5.19) and (5.20) are valid without assuming (5.16). Further, according to Theorem 3.1, the bounds (5.18) and (5.19) are equivalent to the bounds (5.18) and (5.20). Let us denote by $G_\Theta^{out}(\mathbf{A}_0, \cdots, \mathbf{A}_N)$ the set of symmetric matrices $\mathbf{A}^e$ that satisfy (5.18) and (5.19) or (5.20). Let $G_\Theta(\mathbf{A}_0, \cdots, \mathbf{A}_N)$ be the G_Θ-closure of $(\mathbf{A}_0, \cdots, \mathbf{A}_N)$ with volume fraction $(\theta_0, \cdots, \theta_N)$. If no confusion arises, both $G_\Theta^{out}(\mathbf{A}_0, \cdots, \mathbf{A}_N)$ and $G_\Theta(\mathbf{A}_0, \cdots, \mathbf{A}_N)$ are sometimes shortly written as G_Θ^{out} and G_Θ, respectively. Clearly, G_Θ^{out} is a closed and convex set in $I\!R^{n\times n}_{sym}$ and contains G_Θ. Let (cf. (5.2))

$$\begin{cases} \mathcal{S}_{co}^L = \{\mathbf{A}^e \in G_\Theta^{out} : \ \mathrm{Tr}[(\mathbf{A}_0^{-1/2}\mathbf{A}^e\mathbf{A}_0^{-1/2} - \mathbf{I})^{-1}] = \triangle c_*^L\} \\ \mathcal{S}_{co}^U = \{\mathbf{A}^e \in G_\Theta^{out} : \ \mathrm{Tr}[(\mathbf{A}_N^{-1/2}\mathbf{A}^e\mathbf{A}_N^{-1/2} - \mathbf{I})^{-1}] = \triangle c_*^U\} \end{cases} \tag{5.24}$$

be two hypersurfaces in $I\!R^{n\times n}_{sym}$. It is worthwhile noticing that Grabovsky [12] has shown that $G_\Theta^{out} = G_\Theta$ for two-phase well-ordered conductive composites.

We now discuss the attainable and non-attainable points on $\mathcal{S}_{co}^L$ and $\mathcal{S}_{co}^U$. Let $\mathbf{A} \in \mathcal{S}_{co}^L$ and define

$$\mathbf{Q}_i^L = (\mathbf{A}_0^{-1/2}\mathbf{A}\mathbf{A}_0^{-1/2} - \mathbf{I})^{-1} - \frac{1 + \triangle c_*^L}{1 + \triangle c_i^L}(\mathbf{A}_0^{-1/2}\mathbf{A}_i\mathbf{A}_0^{-1/2} - \mathbf{I})^{-1} \tag{5.25}$$

for $i = 1, \cdots, N$. By Theorem 4.1, equations (3.2) and (3.8), we see that $\mathbf{A}$ is attainable if and only if there exists a sequential E-inclusion corresponding to $\mathbb{K}^L = (\mathbf{Q}_1^L, \cdots, \mathbf{Q}_N^L)$ and $\Theta^L = (\theta_1, \cdots, \theta_N)$. Taking the traces of both sides of (5.25), we have

$$\mathrm{Tr}(\mathbf{Q}_i^L) = \frac{\triangle c_*^L - \triangle c_i^L}{(1 + \triangle c_i^L)}. \tag{5.26}$$

From (5.21), it is clear that

$$\triangle c_*^L \geq \min\{\triangle c_i^L : \ i = 1, \cdots, N\} = \triangle c_N^L > 0. \tag{5.27}$$

Thus, $\mathbf{Q}_i^L$ cannot be all negative semi-definite for $i = 1, \cdots, N$ unless $\triangle c_*^L = \triangle c_N^L$. Similarly, let $\mathbf{A} \in \mathcal{S}_{co}^U$ and define

$$\mathbf{Q}_i^U = -(\mathbf{I} - \mathbf{A}_N^{-1/2}\mathbf{A}\mathbf{A}_N^{-1/2})^{-1} + \frac{1 + \triangle c_*^U}{1 + \triangle c_i^U}(\mathbf{I} - \mathbf{A}_N^{-1/2}\mathbf{A}_i\mathbf{A}_N^{-1/2})^{-1} \tag{5.28}$$

for $i = 0, \cdots, N-1$. Then $\mathbf{A}$ is attainable if, and only if there exists a sequential E-inclusion corresponding to $\mathbb{K}^U = (\mathbf{Q}_0^U, \cdots, \mathbf{Q}_{N-1}^U)$ and $\Theta^U = (\theta_0, \cdots, \theta_{N-1})$. Also, $\mathrm{Tr}(\mathbf{Q}_i^U) = \frac{\triangle c_*^U - \triangle c_i^U}{1+\triangle c_i^U}$,

$$\triangle c_*^U \leq \max\{\triangle c_i^U : \ i = 0, \cdots, N-1\} = \triangle c_0^U < -1. \tag{5.29}$$

Thus, $\mathbf{Q}_i^U$ cannot be all negative semi-definite for $i = 0, \cdots, N-1$ unless $\triangle c_*^U = \triangle c_0^U$.

From Theorem 4.1, Theorem 5.2 and Theorem 5.5, we summarize

Theorem 5.7. *Consider conductive composites of* $(N+1)$*-phases with conductivity tensors* $\mathbf{A}_0 < \mathbf{A}_1, \cdots, \mathbf{A}_{N-1} < \mathbf{A}_N$ *and volume fractions* $(\theta_0, \theta_1, \cdots, \theta_{N-1}, \theta_N)$. *Let* $\mathbf{A} \in \mathcal{S}^L_{co}$ *(resp.* $\mathbf{A} \in \mathcal{S}^U_{co}$*) and matrices* $\mathbb{K}^L$ *(resp.* $\mathbb{K}^U$*) be defined as in* (5.25) *(resp.* (5.28)*).*

(i) If $\mathbb{K}^L$ *and* $\Theta^L = (\theta_1, \cdots, \theta_N)$ *(resp.* $\mathbb{K}^U$ *and* $\Theta^U = (\theta_0, \cdots, \theta_{N-1})$*) do not satisfy* (4.1), *then* $\mathbf{A}$ *is not attainable.*
(ii) If the matrices in $\mathbb{K}^L$ *(resp.* $\mathbb{K}^U$*) are all positive semi-definite, then* $\mathbf{A}$ *is attainable.*

Below we specialize Theorem 5.7 to composites of $(N+1)$-isotropic phases of conductivities $k_0, \cdots, k_N$ and volume fractions $\theta_0, \cdots, \theta_N \in [0,1]$. Without loss of generality, we assume $0 < k_0 < k_1 < \cdots < k_{N-1} < k_N$. In terms of $k_0, \cdots, k_N$, we have

$$\begin{cases} \triangle c_i^L = \frac{nk_0}{k_i - k_0} \\ \triangle c_i^U = \frac{nk_N}{k_i - k_N} \end{cases} \quad \text{and} \quad \begin{cases} \triangle c_*^L = \frac{\theta_0 + \sum_{i=1}^N \theta_i \triangle c_i^L/(1+\triangle c_i^L)}{\sum_{i=1}^N \theta_i/(1+\triangle c_i^L)} \\ \triangle c_*^U = \frac{\theta_N + \sum_{i=0}^{N-1} \theta_i \triangle c_i^U/(1+\triangle c_i^U)}{\sum_{i=0}^{N-1} \theta_i/(1+\triangle c_i^U)} \end{cases}.$$

We first show the non-attainable bounds implied by part (i) of Theorem 5.7. Let $\mathbf{A} \in \mathcal{S}^L_{co}$ and denote by $k_1^L, \cdots, k_n^L \in [k_0, k_N]$ the eigenvalues of $\mathbf{A}$. Then the eigenvalues of $\mathbf{Q}_i^L$ are

$$\frac{k_0}{k_1^L - k_0} - \frac{\triangle c_i^L(1+\triangle c_*^L)}{n(1+\triangle c_i^L)}, \cdots, \frac{k_0}{k_n^L - k_0} - \frac{\triangle c_i^L(1+\triangle c_*^L)}{n(1+\triangle c_i^L)} \tag{5.30}$$

for $i = 1, \cdots, N$. Direct calculations reveal that equation (4.1) can be written as

$$\theta_0 \sum_{i=1}^N \frac{\theta_i(1+\triangle c_*^L)}{1+\triangle c_i^L}\left[\frac{k_0}{k_\alpha^L - k_0} - \frac{\triangle c_i^L(1+\triangle c_*^L)}{n(1+\triangle c_i^L)}\right] \geq \tag{5.31}$$
$$\theta_0 \sum_{i=1}^N \theta_i \left[\frac{k_0}{k_\alpha^L - k_0} - \frac{\triangle c_i^L(1+\triangle c_*^L)}{n(1+\triangle c_i^L)}\right]^2 + \left[\sum_{i=1}^N \theta_i \Big(\frac{k_0}{k_\alpha^L - k_0} - \frac{\triangle c_i^L(1+\triangle c_*^L)}{n(1+\triangle c_i^L)}\Big)\right]^2,$$

for all $\alpha = 1, \cdots, n$. From part (i) of Theorem 5.7, we see that $\mathbf{A} \in \mathcal{S}^L_{co}$ is not attainable if their eigenvalues k_α^L do not satisfy (5.31) for one $\alpha \in \{1, \cdots, n\}$. In particular, we notice
If $\theta_0 = 0$, $n \geq 2$, *and at least two of* $\theta_1, \cdots, \theta_N$ *are nonzero, none of the points on* $\mathcal{S}^L_{co}$ *is attainable.*
To show this, we first notice that equation (5.31) implies $k_1^L = k_2^L = \cdots = k_n^L = k_0(1 + n/\triangle c_*^L) =: k_*^L$. Thus, none of the anisotropic points on $\mathcal{S}^L_{co}$ is attainable. Further, if $\mathbf{A} = k_*^L \mathbf{I} \in \mathcal{S}^L_{co}$ is attainable, by Theorem 4.1 we have the existence of sequential E-inclusions $\nu = \sum_{i=1}^N \theta_i \delta_{\mathbf{Q}_i^L}$, where $\mathbf{Q}_i^L$ $(i = 1, \cdots, N)$ are as in (5.25). Since $\mathbf{A}$ is isotropic, all $\mathbf{Q}_i^L$ $(i = 1, \cdots, N)$ are isotropic. Then ν being a gradient Young measure with zero center of

mass implies all $\mathbf{Q}_i^L$ are the zero matrix if $\theta_i \neq 0$ [1]. Taking traces of (5.25) we have $\triangle c_*^L = \triangle c_i^L$ for all $i \in \{1, \cdots, N\}$ such that $\theta_i \neq 0$, which is clearly not possible since at least two of $\theta_1, \cdots, \theta_N$ are nonzero.

Clearly, the above calculations apply to the upper bound, which will not be repeated here. We remark that when $\theta_0 = 0$, the non-attainability of HS bounds for three-phase isotropic composites of isotropic phases has been noticed by Albin, Cherkaev & Nesi [1].

We next show the attainable bounds implied by part (ii) of Theorem 5.7. For $\mathbf{A} \in \mathcal{S}_{co}^L$ (resp. $\mathcal{S}_{co}^U$) with eigenvalues k_α^L (resp . k_α^U) $\in [k_0, k_N]$, matrices in $\mathbb{K}^L$ in (5.25) (resp. $\mathbb{K}^U$ in (5.28)) are all positive semi-definite if, and only if (cf., (5.30))

$$\frac{k_0}{k_\alpha^L - k_0} \leq \frac{\triangle c_1^L(1 + \triangle c_*^L)}{n(1 + \triangle c_1^L)} \qquad \forall\, \alpha = 1, \cdots, n \tag{5.32}$$

$$\Big(\text{resp .} \quad \frac{k_N}{k_\alpha^U - k_N} \leq \frac{\triangle c_{N-1}^U(1 + \triangle c_*^U)}{n(1 + \triangle c_{N-1}^U)} \qquad \forall\, \alpha = 1, \cdots, n\Big). \tag{5.33}$$

From part (ii) of Theorem 5.7, we see that $\mathbf{A} \in \mathcal{S}_{co}^L$ (resp. $\mathcal{S}_{co}^U$) is attainable if their eigenvalues k_α^L (resp. k_α^U) satisfy (5.32) (resp. (5.33)). Taking into account the classic Wiener bounds (5.18), we see that all of $\mathcal{S}_{co}^L$ (resp. $\mathcal{S}_{co}^U$) are attainable if

$$\frac{k_\Theta}{k_0} \leq 1 + \frac{(k_*^L - k_0)(k_1 + (n-1)k_0)}{k_0(k_*^L + (n-1)k_0)} \tag{5.34}$$

$$\Big(\text{resp .} \ \frac{h_\Theta}{k_N} \geq 1 + \frac{(k_*^U - k_N)(k_{N-1} + (n-1)k_N)}{k_N(k_*^U + (n-1)k_N)}\Big), \tag{5.35}$$

where $k_\Theta = \sum_{i=0}^N \theta_i k_i$, $h_\Theta = [\sum_{i=0}^N \theta_i / k_i]^{-1}$, $k_*^L = k_0(1 + n/\triangle c_*^L)$ and $k_*^U = k_N(1 + n/\triangle c_*^U)$. It is worthwhile noticing that both (5.34) and (5.35) hold as equalities for any volume fractions if $N = 1$, while if $N > 1$, satisfaction of (5.34) or (5.35) requires restriction on the volume fractions.

For the isotropic point $(\mathbf{A})_{ij} = k_*^L \delta_{ij}$ on $\mathcal{S}_{co}^L$ (resp. $(\mathbf{A})_{ij} = k_*^U \delta_{ij}$ on $\mathcal{S}_{co}^U$), equation (5.32) (resp. (5.33)) becomes $\triangle c_*^L \geq \triangle c_1^L$ (resp . $\triangle c_*^U \leq \triangle c_{N-1}^U$), i.e.,

$$k_*^L \leq k_1 \qquad \big(\text{resp. } \ k_*^U \geq k_{N-1}^U\big). \tag{5.36}$$

From above discussions, we see that $k_*^L \delta_{ij}$ (resp. $k_*^U \delta_{ij}$) is attainable if equation (5.36) is satisfied. We remark that this attainability result concerning isotropic composites of isotropic materials were first shown by Milton [25].

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[1] This can be shown by considering the 2×2 diagonal minor $X \mapsto m(X)$. From the well-known formula $m(\int X d\nu(X)) = \int m(X) d\nu(X)$, we verify that ν must be a Dirac mass at the zero matrix.

5.4. Composites of elastic materials

We now consider elastic composites of $(N+1)$ phases with elasticity tensors given by $\mathbf{L}_0, \cdots, \mathbf{L}_N$ and volume fractions $\theta_0, \cdots, \theta_N \in [0,1]$. Assume that $\mathbf{L}_0$ and $\mathbf{L}_N$ are isotropic with shear moduli μ_1 and μ_N and that,

$$2\mu_1|\mathbf{F}|^2 < \min_{i\in\{1,\cdots,N\}} \mathbf{F}\cdot\mathbf{L}_i\mathbf{F} \quad \text{and} \quad 2\mu_N|\mathbf{F}|^2 > \max_{i\in\{0,\cdots,N-1\}} \mathbf{F}\cdot\mathbf{L}_i\mathbf{F} \quad (5.37)$$

for all $0 \neq \mathbf{F} \in I\!R^{n\times n}_{sym}$ with $\mathrm{Tr}(\mathbf{F}) = 0$. Let κ_c^L and κ_c^U be the least and greatest number in $\{\mathbf{I}\cdot\mathbf{L}_i\mathbf{I}/n^2 : \ i = 0, \cdots, N\}$, respectively. Also, let $(\mathbf{L}_c^L)_{piqj} = \mu_1(\delta_{ij}\delta_{pq} + \delta_{pj}\delta_{iq}) + (\kappa_c^L - 2\mu_1/n)\delta_{ip}\delta_{jq}$, $(\mathbf{L}_c^U)_{piqj} = \mu_N(\delta_{ij}\delta_{pq} + \delta_{pj}\delta_{iq}) + (\kappa_c^U - 2\mu_N/n)\delta_{ip}\delta_{jq}$, and

$$\begin{cases} \triangle c_i^L = \mathbf{I}\cdot(\mathbf{L}_i - \mathbf{L}_c^L)^{-1}\mathbf{I} \\ \triangle c_i^U = -\mathbf{I}\cdot(\mathbf{L}_c^U - \mathbf{L}_i)^{-1}\mathbf{I} \end{cases} \qquad (i = 0, \cdots, N). \qquad (5.38)$$

With these notations and choosing $\mathbf{L}_c^L$ as the comparison tensor for the lower bound and $\mathbf{L}_c^U$ as the comparison tensor for the upper bound, the HS bounds (3.1) can be written as

$$\mathbf{F}\cdot\mathbf{L}_c^L\mathbf{F} + \mathrm{Tr}(\mathbf{F})^2/\triangle c_*^L \le \mathbf{F}\cdot\mathbf{L}^e\mathbf{F} \le \mathbf{F}\cdot\mathbf{L}_c^U\mathbf{F} + \mathrm{Tr}(\mathbf{F})^2/\triangle c_*^U \quad (5.39)$$

respectively, where $\triangle c_*^L$ (resp. $\triangle c_*^U$) is given by (5.21) (resp. (5.22)) with $\triangle c_i^L$ and $\triangle c_i^U$ given by (5.38), and $\mathbf{L}^e$ denotes the effective tensor of a composite with the prescribed volume fractions. In particular, one shall notice that the lower (resp. upper) bound in (5.39) are valid without assuming $\mathbf{L}_N$ (resp. $\mathbf{L}_0$) being isotropic and the second (resp. first) equation in (5.37).

In general, the discussion of the attainability of a given effective elasticity tensor would be difficult since an inequality in (5.39) holds as an equality for an $\mathbf{F}$ does not fully determine $\mathbf{L}^e$, see (3.2). Nevertheless, we may discuss the attainability of the particular component corresponding to $\mathbf{F}$ of the effective elasticity tensor. By Theorem 4.1 and Theorem 5.5, we know the lower bound in (5.39) is attainable for $\mathbf{F}$ with $\mathrm{Tr}(\mathbf{F}) \neq 0$ if

$$\mathbf{Q}_i^L = \mathbf{F} - \mathrm{Tr}(\mathbf{F})\frac{(1 + n\kappa_c^L\triangle c_*^L)}{\triangle c_*^L(1 + \kappa_c^L\triangle c_i^L)}(\mathbf{L}_i - \mathbf{L}_c^L)^{-1}\mathbf{I} \ \ (i = 1, \cdots, N) \ (5.40)$$

are all negative semi-definite or all positive semi-definite, whereas the upper bound in (5.39) is attainable for $\mathbf{F}$ with $\mathrm{Tr}(\mathbf{F}) \neq 0$ if

$$\mathbf{Q}_i^U = \mathbf{F} - \mathrm{Tr}(\mathbf{F})\frac{(1 + n\kappa_c^U\triangle c_*^U)}{\triangle c_*^U(1 + \kappa_c^U\triangle c_i^U)}(\mathbf{L}_i - \mathbf{L}_c^U)^{-1}\mathbf{I} \ \ (i = 0, \cdots, N-1) \ (5.41)$$

are all negative semi-definite or all positive semi-definite.

Denote by $\kappa_\Theta^e = \mathbf{I}\cdot\mathbf{L}^e\mathbf{I}/n^2$ the bulk moduli of the composite. Choosing $\mathbf{F} = \mathbf{I}$ in (5.39) we obtain

$$\kappa_c^L + 1/\triangle c_*^L \le \kappa_\Theta^e \le \kappa_c^U + 1/\triangle c_*^U, \qquad (5.42)$$

which coincides with the Walpole's bounds (WALPOLE [38]) for bulk moduli. If we further assume all phases $\mathbf{L}_0, \cdots, \mathbf{L}_N$ are isotropic tensors, all $(\mathbf{Q}_1^L, \cdots, \mathbf{Q}_N^L)$ in (5.40) (resp. all $(\mathbf{Q}_1^U, \cdots, \mathbf{Q}_N^U)$ in (5.41)) are either negative semi-definite or positive semi-definite is equivalent to

$$\triangle c_*^L \geq \max_{i \in \{1, \cdots, N\}} \triangle c_i^L \text{ or } \triangle c_*^L \leq \min_{i \in \{1, \cdots, N\}} \triangle c_i^L \tag{5.43}$$

$$\big(\text{resp. } \triangle c_*^U \leq \min_{i \in \{0, \cdots, N-1\}} \triangle c_i^U \text{ or } \triangle c_*^U \geq \max_{i \in \{0, \cdots, N-1\}} \triangle c_i^U\big). \tag{5.44}$$

Thus, the lower bound in (5.42) is attainable if (5.43) is satisfied while the upper bound in (5.42) is attainable if (5.44) is satisfied. These attainability results concerning bulk moduli of isotropic phases implies what were previously obtained by MILTON [25].

6. Summary and discussions

We have derived a necessary and sufficient condition for the HS bounds (cf., (2.37) and (3.1)) to be attainable. This condition renders a simple characterization of the gradient fields of optimal structures/microstructures motivates us to introduce the concept of sequential E-inclusions. A special quasiconvex function is used to restrict sequential E-inclusions, while a convex property of gradient Young measures is used to show the existence of a class of sequential E-inclusions. From these results, we find an outer bound and inner bound on the attainable HS bounds for composites of any finite number of conductive materials or elastic materials in any dimensions.

We have restricted ourselves to periodic composites for the ease of the definition of the effective tensors (cf. (2.2)) and the formal proofs. Since any effective tensors can be approximated arbitrarily well by those of periodic structures [2], the results in this paper, in particular, Theorem 3.1, Corollary 3.2, Theorem 4.1, and all results in Section 5 remain valid without assuming the composites are periodic.

Since the G-closure of two well-ordered conductive materials can be realized by multi-rank laminations (LURIE & CHERKAEV [23]; TARTAR [36]; GRABOVSKY [12]), by Theorem 4.1 we know that sequential E-inclusions in Theorem 5.4 can all be realized by multi-rank laminations. Further, it is sufficient to consider simple laminations to prove Theorem 5.3, see KINDERLEHRER & PEDREGAL [19]. From these two facts we can infer that sequential E-inclusions in Theorem 5.5, and therefore all attainable HS bounds in Theorem 5.6, can be realized by multi-rank laminations. A formal proof of this statement is not pursued here.

The use of the convexity property of gradient Young measures (cf., Theorem 5.3) eases the constructions of sequential E-inclusions on one hand,

[2] This result is attributed to DAL MASO & KOHN in the literature, see e.g. ALLAIRE [3].

and on the other hand, to establish the existence of a sequential E-inclusion that is outside what is covered by Theorem 5.5, we have to come back to the conventional way of constructions. There are three types of constructions of which we are aware that can give rise to sequential E-inclusions beyond Theorem 5.5:

(i) In the case of two dimensions ($n = 2$) and three phases ($N = 2$), the Sigmund's construction (SIGMUND [30]; GIBIANSKY & SIGMUND [10]) in effect asserts the existence of sequential E-inclusions corresponding to $\mathbb{K} = (p_1\mathbf{I}/2, p_2\mathbf{I}/2)$ and $\Theta = (\theta_1, \theta_2)$ if $\mathbb{K}$ and Θ satisfy

$$\frac{1}{\sqrt{\theta_1}} + \frac{p_1}{p_2} \geq 1. \tag{6.1}$$

To see this, one notices equation (80) in Gibiansky and Sigmund (2000) can be rewritten as

$$\theta_0 \geq \frac{(\sqrt{\theta_1} - \theta_1)(\triangle c_1^L - \triangle c_2^L)}{\triangle c_1^L + 1}. \tag{6.2}$$

By Theorem 4.1, (5.21) and (5.26), we find

$$\frac{p_1}{p_2} = \frac{(1 + \triangle c_2^L)(\triangle c_*^L - \triangle c_1^L)}{(1 + \triangle c_1^L)(\triangle c_*^L - \triangle c_2^L)} = \frac{\theta_0(1 + \triangle c_2^L) + \theta_2(\triangle c_2^L - \triangle c_1^L)}{\theta_0(1 + \triangle c_1^L) + \theta_1(\triangle c_1^L - \triangle c_2^L)}. \tag{6.3}$$

Plugging (6.3) into (6.2) and eliminating $\triangle c_1^L$, $\triangle c_2^L$, one can show equation (6.2) is equivalent to (6.1).

(ii) In the case of two dimensions ($n = 2$) and three phases ($N = 2$), the constructions of ALBIN, CHERKAEV & NESI [1] assert the existence of sequential E-inclusions outside Theorem 5.5. However, we do not have a simple formula on $\mathbb{K}$ and Θ associated with sequential E-inclusions that can be realized in this and following constructions.

(iii) In two and higher dimensions, $N \geq 2$, periodic E-inclusions can give rise to sequential E-inclusions that have Dirac masses supported on both negative definite matrices and positive definite matrices, see LIU, JAMES & LEO [22].

All the above constructions could be important in extending the attainable HS bounds. A systematic study is underway and will be reported in the future.

A composite being optimal often implies some unusual property, say, curl-free, of the solution of equation (2.3). We now give a second viewpoint of this property for the case of lower bound. By Holmholtz decomposition we can write a solution $\mathbf{u_F} \in W^{1,2}_{per}(Y, \mathbb{R}^n)$ of equation (2.3) as

$$\mathbf{u_F} = -\nabla\xi + \mathbf{w},$$

where $\xi \in W^{2,2}_{per}(Y)$, $\mathbf{w} \in W^{1,2}_{per}(Y, I\!R^n)$ and $\operatorname{div} \mathbf{w} = 0$. With a given comparison material $\mathbf{L}_c$, we have

$$\begin{aligned}\mathbf{F} \cdot \mathbf{L}^e(\mathcal{O})\mathbf{F} &= \fint_Y (-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F}) \cdot \mathbf{L}(\mathbf{x}, \mathcal{O})(-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F})d\mathbf{x} \\ &= \fint_Y (-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F}) \cdot (\mathbf{L}(\mathbf{x}, \mathcal{O}) - \mathbf{L}_c)(-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F})d\mathbf{x} \\ &+ \fint_Y (-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F}) \cdot \mathbf{L}_c(-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F})d\mathbf{x}. \qquad (6.4)\end{aligned}$$

Again we require tensors $(\mathbf{L}_c, \mathbf{L}_0, \cdots, \mathbf{L}_N)$ are as in Theorem 3.2. If $\mathbf{L}(\mathbf{x}, \mathcal{O}) \geq \mathbf{L}_c$, the first term on the right-hand side of (6.4), denoted by J_1, can be bounded by

$$\begin{aligned}J_1 &= \sum_{i=1}^N \theta_i \fint_{\Omega_i} (-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F}) \cdot (\mathbf{L}_i - \mathbf{L}_c)(-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F})d\mathbf{x} \\ &+ \theta_0 \fint_{\Omega_0} (-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F}) \cdot (\mathbf{L}_0 - \mathbf{L}_c)(-\nabla\nabla\xi + \nabla\mathbf{w} + \mathbf{F})d\mathbf{x} \\ &\geq \sum_{i=1}^N \theta_i(-\mathbf{Q}_i + \mathbf{F}) \cdot (\mathbf{L}_i - \mathbf{L}_c)(-\mathbf{Q}_i + \mathbf{F}) + \frac{\theta_0}{\triangle c_0}[-p_0 + \operatorname{Tr}(\mathbf{F})]^2, \quad (6.5)\end{aligned}$$

where $\mathbf{Q}_i = \fint_{\Omega_i} (\nabla\nabla\xi - \nabla\mathbf{w})d\mathbf{x}$ $(i = 1, \cdots, N)$, and $p_0 \in I\!R$ be such that

$$\sum_{i=1}^N \theta_i \operatorname{Tr}(\mathbf{Q}_i) + p_0\theta_0 = 0. \qquad (6.6)$$

In particular the convexity of mappings $X \mapsto X \cdot (\mathbf{L}_i - \mathbf{L}_c)X$ has been used to bound the integrals over $\Omega_1, \cdots, \Omega_N$, and the fact that $(\mathbf{L}_0 - \mathbf{L}_c)X = \frac{\operatorname{Tr}(X)}{\triangle c_0}\mathbf{I}$ has been used to calculate the integral over Ω_0. The second term on the right-hand side of (6.4), denoted by J_2, can be estimated by

$$\begin{aligned}J_2 &= \fint_Y \mu_1^c |-\nabla\nabla\xi + \nabla\mathbf{w}|^2 + \mu_2^c \operatorname{Tr}[(-\nabla\nabla\xi + \nabla\mathbf{w})^2] + \lambda^c |\Delta\xi|^2 d\mathbf{x} + \mathbf{F} \cdot \mathbf{L}_c\mathbf{F} \\ &\geq \fint_Y (\mu_1^c + \mu_2^c)|\nabla\nabla\xi|^2 + \lambda^c|\Delta\xi|^2 d\mathbf{x} + \mathbf{F} \cdot \mathbf{L}_c\mathbf{F} \qquad (6.7) \\ &= k_c \fint_Y |\Delta\xi|^2 d\mathbf{x} + \mathbf{F} \cdot \mathbf{L}_c\mathbf{F} \\ &\geq k_c[\sum_{i=1}^N \theta_i \operatorname{Tr}(\mathbf{Q}_i)^2 + \theta_0 p_0^2] + \mathbf{F} \cdot \mathbf{L}_c\mathbf{F},\end{aligned}$$

where we have used the following facts:

(i) $\fint_Y \nabla\nabla\xi \cdot \nabla\mathbf{w} d\mathbf{x} = 0$ and $\fint_Y \operatorname{Tr}[(\nabla\mathbf{w})^2] = 0$, which follow from $\operatorname{div} \mathbf{w} = 0$;
(ii) $\fint_Y |\nabla\nabla\xi|^2 d\mathbf{x} = \fint_Y |\Delta\xi|^2 d\mathbf{x}$, which follows from the divergence theorem;
(iii) $\mu_1^c \geq 0$, $\mu_1^c + \mu_2^c \geq 0$ and $k_c = \mu_1^c + \mu_2^c + \lambda^c \geq 0$.

Upon minimizing $J_1 + J_2$ over all matrices $(\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ for fixed $\mathbf{F}$ with constraint (6.6), we find the minimizing $(\mathbf{Q}_1, \cdots, \mathbf{Q}_N)$ satisfy (3.8) and the minimum is $\mathbf{F} \cdot \mathbf{L}_c\mathbf{F} + \mathrm{Tr}(\mathbf{F})^2/\triangle c_*$. By equations (6.4), (6.5) and (6.7), we therefore obtain the lower bound in (3.1). Checking the conditions for the inequalities in (6.5) and (6.7) to hold as equalities, one can see that the bound is attained if, and only if

$$\begin{cases} \mu_1^c \fint_Y |\nabla \mathbf{w}|^2 d\mathbf{x} = 0 & \text{on } Y \\ \Delta\xi = p_0 & \text{on } \Omega_0 \\ \nabla\nabla\xi - \nabla\mathbf{w} = \mathbf{Q}_i & \text{on } \Omega_i, \ i = 1, \cdots, N \end{cases} . \tag{6.8}$$

Clearly, at least one of these three conditions in (6.8) will be violated by the field of an optimal composite when the lower bound in (3.1) becomes not attainable. The above calculation in effect is a new proof of Theorem 3.2 for the case of lower bound. More importantly, it provides us the possibilities of violations of (6.8), and hence suggests that tighter bounds than (3.1) would be unlikely without incorporating these violations into the relaxation arguments, see recent work of ALBIN, CONTI & NESI [2] on progress of improved bounds for two-dimensional three-phase conductive composites.

From the viewpoint of translation method, we are using $\mathbf{L}_c$ as the "translation" tensor in the above calculations. Although the mapping $\mathbf{F} \mapsto \mathbf{F} \cdot \mathbf{L}_c\mathbf{F}$ is not a null Lagrangian, we can still find a good lower bound for the second term on the right-hand side of (6.4), see (6.7). Note that the above calculations only require the constants μ_1^c, μ_2^c, λ^c satisfy

$$\mu_1^c \geq 0, \ \ \mu_1^c + \mu_2^c \geq 0 \quad \text{and} \quad k_c = \lambda^c + \mu_1^c + \mu_2^c \geq 0.$$

The consequences of such general translation tensors are worthwhile being explored in other problems, say, the relaxation of multi-well energies, see KOHN [21]; SMYSHLYAEV & WILLIS [33]; and CHENCHIAH & BHATTACHARYA [8].

Finally, we make a few comments on directions of generalization of results in this paper. First of all, one notices that the key step in Section 2 is that the solution of equation (2.12) can be identified as a linear combination of the gradients of the solution of the simple scalar equation (2.24). So, as long as this is guaranteed, the restriction of $m = n$ and the polarization being piecewise dilatational can both be removed (cf., (2.27)). In this way one can derive the cross-property bounds (BERGMAN [7]; SILVESTRE [31]) and address the problem of optimizing both the thermal conductivity and electric conductivity (TORQUATO, HYUN & DONEV [37]).

Further, the restriction on the comparison material, i.e., $\mathbf{L}_c$ is of form (3.6)), can be relaxed considerably by a linear transformation

$$\mathbf{x} \longrightarrow \mathbf{x}' = \Lambda^{-1}\mathbf{x} \qquad \text{and} \qquad \mathbf{v} \longrightarrow \mathbf{v}' = \mathbf{G}^{-1}\mathbf{v}, \tag{6.9}$$

where $\mathbf{G}, \Lambda \in \mathbb{R}^{n\times n}$ are invertible. Therefore, the preceding results can be extended to all comparison tensors (while other conditions on material properties in Theorem 3.2 remain unchanged)

$$(\mathbf{L}'_c)_{piqj} = (\mathbf{G})_{rp}(\mathbf{G})_{sq}(\Lambda)_{ik}(\Lambda)_{jl}(\mathbf{L}_c)_{rksl},$$

where $\mathbf{L}_c$ is of form (3.6). In fact we have used the transformations (6.9) with $\Lambda = \mathbf{A}_0^{-1/2}$ (or $\Lambda = \mathbf{A}_N^{-1/2}$) and $\mathbf{G} = \mathbf{I}$ in writing the bounds as (5.19) and (5.20). Further, through a refined calculation, we can extend the comparison tensors to those satisfy $(m = n)$

$$(\mathbf{L}_c)_{piqj}(\mathbf{k})_i(\mathbf{k})_j(\mathbf{k})_q = k_c|\mathbf{k}|^2(\mathbf{k})_p \qquad \forall\, \mathbf{k} \in I\!R^n \tag{6.10}$$

for some $k_c > 0$, see details in Liu, James & Leo [22]. The linear transformation (6.9) can be again applied to general $\mathbf{L}_c$ of form (6.10) and further extend the applicability of the preceding results. The reader is invited to formulate the precise statements corresponding to Theorem 3.2 and Theorem 4.1 for tensors $\mathbf{L}_c$ of these general forms.

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Liping Liu
Division of Engineering and Applied Science
California Institute of Technology
Pasadena, CA 91125
email:liuliping@caltech.edu